

p -ADIC FRAMED BRAIDS II

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ABSTRACT. In this paper, which is sequel to [4], we explore further the structures of the p -adic framed braids and the p -adic Yokonuma–Hecke algebras constructed in [4], by means of dense sub-structures approximating p -adic elements. We then construct a p -adic Markov trace on the p -adic Yokonuma–Hecke algebras, which arises naturally as the inverse limit of classical Markov traces constructed in [3], and we approximate the values of the p -adic trace on p -adic elements.

Surprisingly, the traces in [3] do not normalize directly to yield isotopy invariants of oriented framed links. This leads to imposing the ‘ E -condition’ to the trace parameters. For solutions of the ‘ E -system’ we then define \mathbb{C} -valued isotopy invariants of oriented framed links, which lift to isotopy invariants of p -adic framed links.

The Yokonuma–Hecke algebras have topological interpretations in the context of framed braids, of singular braids and of classical braids.

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INTRODUCTION

In [4] we constructed the p -adic framed braids, $\mathcal{F}_{p^\infty, n}$, and the p -adic Yokonuma–Hecke algebras $Y_{p^\infty, n}(u)$. The group $\mathcal{F}_{p^\infty, n}$ is defined as the inverse limit $\varprojlim_r \mathcal{F}_{p^r, n}$ of modular framed braid groups, so a p -adic framed braid may be viewed as a sequence of the same classical braid with framings of the corresponding strands forming a p -adic integer. By certain group isomorphisms, a p -adic framed braid may also be viewed as a classical braid with framings p -adic integers or as a classical framed braid, but with infinite cablings replacing each strand and corresponding framings forming a p -adic integer. View Figure 1 for the different facets of a p -adic framed braid. In $\mathcal{F}_{p^\infty, n}$ there are no modular relations and in the heart of $\mathcal{F}_{p^\infty, n}$ lies a dense copy of the classical framed braid group \mathcal{F}_n .

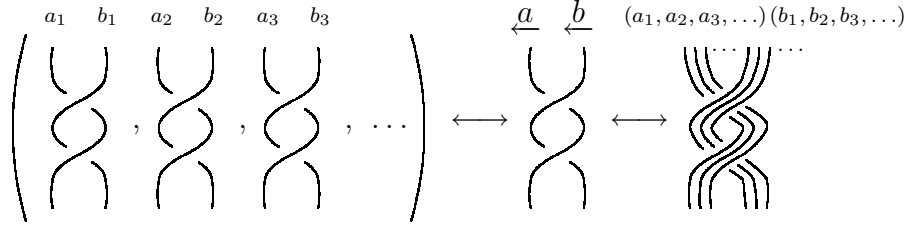


FIGURE 1. The different facets of a p -adic framed braid

The classical Yokonuma–Hecke algebra $Y_{d,n}(u)$, cf. [10], is a quotient of the modular framed braid group $\mathcal{F}_{d,n}$ over a quadratic relation, see (1.4), involving the framing generators t_i in a subtle way, by means of certain weighted idempotents $e_{d,i}$, see (1.3). View Figures 3 and 4 for some diagrammatic interpretations and also [4] for more. Setting $d = 1$ the algebra $Y_{d,n}(u)$ coincides with the classical Iwahori–Hecke algebra.

The p -adic Yokonuma–Hecke algebra $Y_{p^\infty,n}(u)$ arises as the inverse limit of the algebras $Y_{p^r,n}(u)$. In $Y_{p^\infty,n}(u)$ there are no modular relations. Yet, a similar quadratic relation holds also in $Y_{p^\infty,n}(u)$, see (1.14). Moreover, the elements $e_{p^\infty,i}$ (see 1.12), lifts of the idempotents $e_{p^r,i}$, are still idempotents in $Y_{p^\infty,n}(u)$ but no more weighted sums. They are truly p -adic elements and they can be interpreted as infinite series, approximated by the elements $e_{p^r,i}$. In the heart of $Y_{p^\infty,n}(u)$ lies a dense subalgebra, quotient of \mathcal{F}_n (see Theorem 1).

Further, the first author constructed in [3] a linear Markov trace on the classical Yokonuma–Hecke algebra $Y_{d,n}(u)$. In this paper we extend the construction to a p -adic Markov trace on the algebra $Y_{p^\infty,n}(u)$. It arises as the inverse limit of the traces in [3], that depends on the parameter d . The p -adic trace takes values in the inverse limit of certain polynomial rings. We approximate the trace of a p -adic element by constant sequences of polynomials. Note that, restricting to the dense subalgebras, the p -adic Markov trace defines a representation of classical framed braids.

The Yokonuma–Hecke algebras are very versatile algebraic objects, in the sense that they can be used for completely different topological interpretations. They comprise the only examples we know of having this property. Indeed, apart from the framed braids, they are also related to classical and singular braids, as there is a monoid representation from the singular braid monoid algebra to $Y_{d,n}(u)$ (not onto). Then, the traces in [3] are also Markov traces on the singular braid monoid. See [5] for details.

From the topological point of view, closing a framed braid gives rise to an oriented framed link and closing a p -adic framed braid gives rise to an oriented p -adic framed link, view Figure 2. Further, closing a classical or a singular

braid gives rise to an oriented classical or singular link. So, the next consideration is to try to re-scale, if necessary, and normalize the traces in [3] and also the p -adic trace, in order to obtain isotopy invariants of oriented framed links or oriented classical and singular links, according to the corresponding Markov braid theorems.

Trying to do this leads to imposing the ‘ E -condition’ on the trace parameters. The traces in [3] are the only Markov traces we know of, which do not normalize directly to yield isotopy invariants of knots. In this paper we explain the ‘ E -condition’ and we explore the corresponding ‘ E -system’. Surprisingly, there are always non-trivial solutions of the E -system in the set of complex numbers (see Appendix). Given now the E -condition, we normalize the traces in [3] and the p -adic trace and we define isotopy invariants of oriented framed links. These invariants lift to a p -adic invariant of p -adic and classical framed links.

Similarly, given the E -condition, we defined in [5] invariants of oriented singular links, using the monoid representation from the singular braid monoid to the Yokonuma–Hecke algebra and the Markov braid theorem for singular braids. Analogously, in [6] we defined invariants of classical oriented links.

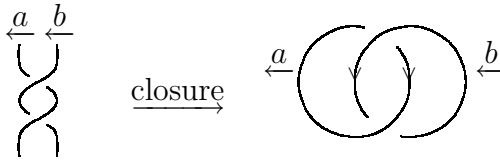


FIGURE 2. A p -adic framed braid and a p -adic framed link

It is well-known that framed links can be used for constructing (closed, connected, oriented) 3-manifolds using topological surgery. Then two such manifolds are homeomorphic if and only if any two framed links in S^3 representing them are related through isotopy and the Kirby moves, or the equivalent Fenn–Rourke moves [1]. So, isotopy invariants of framed links can be used (and have been used) for constructing topological invariants of 3-manifolds. One reason for extending the definition of the Yokonuma–Hecke algebras to the p -adic Yokonuma–Hecke algebra was that we wanted to keep the framing in \mathbb{Z} , not modular, since in the main Kirby move the framings add up.

This paper is a sequel to [4], but we tried to keep it self-contained. We first discuss dense subsets in our p -adic structures. We then use the dense substructures for finding in the group $\mathcal{F}_{p^\infty, n}$ and in the algebras $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ and $Y_{p^\infty, n}(u)$ approximations of p -adic elements by sequences of constant sequences. Further, we use the approximations for interpreting a p -adic element as an infinite series. In [4] we also discussed approximations but here we really

expand on the subject. We also ignore here the identifications related to Figure 1, which we had to observe closely in [4]. Approximating p -adic elements is particularly useful for understanding deeper the p -adic structures as well as for the purpose of p -adic trace computations.

In the course of this study a number of interesting questions arose about the dense substructures. We answered some but many remain open for further investigation. Moreover, it is not known to us at this point of writing whether our invariants for framed links, constructed in this paper, or our invariants for classical links, constructed in [6], or our invariants for singular links, constructed in [5], all deriving from the Yokonuma–Hecke algebras, are distinct from known ones. The Yokonuma–Hecke algebras have a rich structure and the p -adic Yokonuma–Hecke algebra has an even richer structure. The Markov trace on the algebra $Y_{d,n}(u)$ depends on d indeterminates, but after imposing the E -condition we are left with only one indeterminate. We note that in the case of classical knots and links, a ‘closed’ cubic relation satisfied in the Yokonuma–Hecke algebra seems to be more appropriate, for details see [6]. It would be very interesting if our invariants would lead to new 3-manifold invariants and we hope that this new concept of p -adic framed braids and p -adic framed links that we propose will serve to such purposes. Finally, and beyond the above, we thank Sergei Chmutov for reminding us that our traces are more naturally adapted to the context of contact structures.

The paper is organized as follows: In Section 1 we recall briefly our constructions in [4] of the p -adic framed braids and the p -adic Yokonuma–Hecke algebras, giving emphasis to the relations and the main properties in each structure. Results stated as lemmas or propositions are not contained in [4]. For details and proofs of previous results we refer the reader to [4]. In Section 2 we discuss approximations of the various p -adic objects appearing in this work. In Section 3 we recall the Markov traces in [3], we construct our p -adic Markov trace and we give some computations and approximations of the values of the p -adic trace. In Section 4 we give and discuss our E -condition, which is needed for the normalization of the traces in [3]. Normalization of the traces yields an infinite family of oriented framed link invariants, which extends to an invariant for p -adic oriented framed links. These are all to be found in Section 5. We also give computations on concrete examples.

We would like to thank Drossos Gintides and Johannes Grassberger for helping us find interesting solutions to the E -system using mathematical software. Last, we are thankful to Paul Gérardin for computing the general solution of the E -system. We present his elegant proof in the Appendix to this paper.

1. FRAMED BRAIDS, QUOTIENT ALGEBRAS AND p -ADIC OBJECTS

1.1. *Framed braid groups.* The classical braid group on n strands, B_n , is generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$, where σ_i is the positive crossing

between the i th and the $(i + 1)$ st strand, satisfying the well-known braid relations. On the other hand the group \mathbb{Z}^n is generated by the ‘elementary framings’ $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i th position. In the multiplicative notation an element $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ can be expressed as $a = t_1^{a_1} \dots t_n^{a_n}$, where t_1, \dots, t_n are the standard multiplicative generators of \mathbb{Z}^n . The *framed braid group* on n strands is then defined as:

$$(1.1) \quad \mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$$

where the action of B_n on \mathbb{Z}^n is given by the permutation induced by a braid on the indices: $\sigma_i t_j = t_{\sigma_i(j)} \sigma_i$. A word w in \mathcal{F}_n has, thus, the ‘splitting property’, i.e. it splits into the ‘framing’ part and the ‘braiding’ part: $w = t_1^{a_1} \dots t_n^{a_n} \sigma$, where $\sigma \in B_n$. So w is a classical braid with an integer –its framing– attached to each strand. Especially, an element of \mathbb{Z}^n is identified with a framed identity braid on n strands, while a classical braid in B_n is viewed as a framed braid with all framings 0. The multiplication in \mathcal{F}_n is defined by placing one braid on top of the other and collecting the total framing of each strand to the top. For a good treatment of the group \mathcal{F}_n see, for example, [7].

Further, for a positive integer d , the d -*modular framed braid group* on n strands, $\mathcal{F}_{d,n}$, is defined as the quotient of \mathcal{F}_n over the *modular relations*:

$$(1.2) \quad t_i^d = 1 \quad (i = 1, \dots, n)$$

Thus, $\mathcal{F}_{d,n} = (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$. Framed braids in $\mathcal{F}_{d,n}$ have framings modulo d .

1.2. Yokonuma–Hecke algebras. Passing now to the group algebra $\mathbb{C}\mathcal{F}_{d,n}$, we have the following elements, which are idempotents.

$$(1.3) \quad e_{d,i} := \frac{1}{d} \sum_{m=0}^{d-1} t_i^m t_{i+1}^{-m} \quad (i = 1, \dots, n-1)$$

In fact $e_{d,i} \in \mathbb{C}(\mathbb{Z}/d\mathbb{Z})^n$. For a diagrammatic interpretation of $e_{d,1} \in \mathbb{C}\mathcal{F}_{d,3}$ view Figure 3.

$$e_{d,1} = \frac{1}{d} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \text{strand 1} & \text{strand 2} & \text{strand 3} \end{array} + \begin{array}{c|c|c} 1 & d-1 & 0 \\ \hline \text{strand 1} & \text{strand 2} & \text{strand 3} \end{array} + \begin{array}{c|c|c} 1 & d-2 & 0 \\ \hline \text{strand 1} & \text{strand 2} & \text{strand 3} \end{array} + \dots + \begin{array}{c|c|c} d-1 & 1 & 0 \\ \hline \text{strand 1} & \text{strand 2} & \text{strand 3} \end{array} \right)$$

FIGURE 3. The element $e_{d,1} \in \mathbb{C}\mathcal{F}_{d,3}$

In the following we fix a $u \in \mathbb{C} \setminus \{0, 1\}$ and we correspond σ_i to g_i . The *Yokonuma–Hecke algebra* $Y_{d,n}(u)$ is defined as the quotient of the group algebra $\mathbb{C}\mathcal{F}_{d,n}$ over the ideal $I_{d,n}$ generated by the expressions $\sigma_i^2 - 1 - (u-1)e_{d,i} + (u-1)e_{d,i}\sigma_i$, which give rise to the following quadratic relations:

$$(1.4) \quad g_i^2 = 1 + (u-1)e_{d,i} - (u-1)e_{d,i}g_i$$

(see [4] for diagrammatic interpretations). Since the quadratic relation does not change the framing, we have $(\mathbb{Z}/d\mathbb{Z})^n \subset Y_{d,n}(u)$ and we keep the same notation for the elements of $\mathbb{C}(\mathbb{Z}/d\mathbb{Z})^n$ in $Y_{d,n}(u)$. In particular we use the same notation for the elements $e_{d,i}$ in $Y_{d,n}(u)$. The elements g_i are invertible (view Figure 4 for a diagrammatic interpretation):

$$(1.5) \quad g_i^{-1} = g_i - (u^{-1} - 1) e_{d,i} + (u^{-1} - 1) e_{d,i} g_i$$

$$\begin{aligned} & \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) = \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) - \frac{u^{-1}-1}{d} \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) + \frac{u^{-1}-1}{d} \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) \\ & \quad - \frac{u^{-1}-1}{d} \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) + \frac{u^{-1}-1}{d} \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) + \frac{u^{-1}-1}{d} \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) + \cdots + \frac{u^{-1}-1}{d} \left(\begin{array}{c} 0 \\ \text{strand} \end{array} \right) \end{aligned}$$

FIGURE 4. The element $g_1^{-1} \in Y_{d,3}(u)$

From the above, a presentation of $Y_{d,n}(u)$ is given by the generators $t_1, \dots, t_n, g_1, \dots, g_{n-1}$, satisfying: the braid relations and the quadratic relations (1.4) for the g_i 's, the modular relations (1.2) and commuting relations for the t_j 's, together with the mixed relations below, according to (1.1):

$$\begin{aligned} g_i t_i &= t_{i+1} g_i \\ g_i t_{i+1} &= t_i g_i \\ g_i t_j &= t_j g_i \quad \text{for } j \neq i, i+1 \end{aligned}$$

Note that, omitting the quadratic relations (1.4), we have a presentation for $\mathcal{F}_{d,n}$.

Remark 1. There is an epimorphism of the Yokonuma–Hecke $Y_{d,n}(u)$ onto the Iwahori–Hecke algebra $H_n(q)$ via the map

$$\begin{aligned} g_i &\mapsto (1 - q^{-1}) - q^{-1} T_i \\ t_j &\mapsto 1 \end{aligned}$$

where T_i are the standard generators of $H_n(q)$. Further, for $d = 1$ we have all $t_j = 1$ and $Y_{1,n}(u)$ coincides with the algebra $H_n(u)$. Of course, the mapping $g_i \mapsto (i, i+1)$ and $t_j \mapsto 1$ defines an epimorphism of $Y_{d,n}(1)$ onto the group algebra of the symmetric group.

In $Y_{d,n}(u)$ the following relations hold (see Lemma 4, Proposition 5[4]):

$$(1.6) \quad \begin{aligned} t_j e_{d,i} &= e_{d,i} t_j \\ e_{d,j} e_{d,i} &= e_{d,i} e_{d,j} \\ g_j e_{d,i} &= e_{d,i} g_j && \text{for } j \neq i-1, i+1 \\ g_{i-1} e_{d,i} &= e_{d,i-1,i+1} g_{i-1} && \text{and } e_{d,i} g_{i-1} = g_{i-1} e_{d,i-1,i+1} \\ g_{i+1} e_{d,i} &= e_{d,i,i+2} g_{i+1} && \text{and } e_{d,i} g_{i+1} = g_{i+1} e_{d,i,i+2} \end{aligned}$$

where, $e_{d,i,k} = \frac{1}{d} \sum_{1 \leq s \leq d} t_i^s t_k^{-s}$ for any i, k with $k \neq i$, abbreviating $e_{d,i,i+1}$ to $e_{d,i}$. Note that, using (1.5), relations (1.6) are also valid if all g_k 's are replaced by their inverses g_k^{-1} . Clearly $e_{d,i,k} = e_{d,k,i}$ and $e_{d,i,k}^2 = e_{d,i,k}$. Moreover, the following relations hold in $Y_{d,n}(u)$, and can be easily deduced by induction.

Lemma 1. *Let $m \in \mathbb{Z}, k \in \mathbb{N}$. We have:*

- (1) *For m positive, define $\alpha_m = (u-1) \sum_{l=0}^{k-1} u^{2l}$ if $m = 2k$ and $\beta_m = u(u-1) \sum_{l=0}^{k-1} u^{2l}$ if $m = 2k+1$. Then:*

$$g_i^m = \begin{cases} 1 + \alpha_m e_{d,i} - \alpha_m e_{d,i} g_i & \text{if } m = 2k \\ g_i - \beta_m e_{d,i} + \beta_m e_{d,i} g_i & \text{if } m = 2k+1 \end{cases}$$

- (2) *For m negative, define $\alpha'_m = u^{-1}(u^{-1}-1) \sum_{l=0}^{k-1} u^{-2l}$ if $m = -2k$ and $\beta'_m = (u^{-1}-1) \sum_{l=0}^{k-1} u^{-2l}$ if $m = -2k+1$. Then:*

$$g_i^m = \begin{cases} 1 + \alpha'_m e_{d,i} - \alpha'_m e_{d,i} g_i & \text{if } m = -2k \\ g_i - \beta'_m e_{d,i} + \beta'_m e_{d,i} g_i & \text{if } m = -2k+1 \end{cases}$$

1.3. *Inverse limits and the p -adic integers.* Our references for inverse limits are mainly [8] and [9].

An *inverse system* (X_i, ϕ_j^i) of topological spaces indexed by a directed set I , consists of a family $(X_i ; i \in I)$ of topological spaces (groups, rings, algebras, et cetera) and a family $(\phi_j^i : X_i \longrightarrow X_j ; i, j \in I, i \geq j)$ of continuous homomorphisms, such that

$$\phi_i^i = \text{id}_{X_i} \quad \text{and} \quad \phi_k^j \circ \phi_j^i = \phi_k^i \quad \text{whenever } i \geq j \geq k$$

The maps ϕ_j^i are also called *connecting homomorphisms*. If no other topology is specified on the sets X_i they are regarded as topological spaces with the discrete topology. In particular, finite sets are compact Hausdorff spaces. The *inverse limit* $\varprojlim X_i$ of the inverse system (X_i, ϕ_j^i) is defined as:

$$\varprojlim X_i := \{z \in \prod X_i ; (\phi_j^i \circ \varpi_i)(z) = \varpi_j(z) \quad \text{whenever } j \geq i\}$$

where the map ϖ_i denotes the natural projection of $\prod X_i$ onto X_i . Recall that, if $X_i = X$ for all i and ϕ_j^i is the identity for all i, j then $\varprojlim X$ can be identified naturally with X (identifying a constant sequence (x, x, \dots) with $x \in X$).

Notation. In the following we fix a prime number p and we denote by \mathbb{N} the set of positive integers regarded as a directed set with the usual order. Finally, for $r \geq s$ we denote ϑ_s^r the natural epimorphism:

$$(1.7) \quad \begin{array}{ccc} \vartheta_s^r : \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & \mathbb{Z}/p^s\mathbb{Z} \\ m & \mapsto & m \pmod{p^s} \end{array}$$

We denote $C_r = \langle t_r ; t_r^{p^r} = 1 \rangle \cong \mathbb{Z}/p^r\mathbb{Z}$, the cyclic group of order p^r in the multiplicative notation. For $r \geq s$ we denote θ_s^r the following natural connecting epimorphism,

$$(1.8) \quad \begin{array}{ccc} \theta_s^r : C_r & \longrightarrow & C_s \\ t_r^m & \mapsto & t_s^{\vartheta_s^r(m)} \end{array}$$

Thus we obtain an inverse system of groups (C_r, θ_s^r) , whose inverse limit is the group of p -adic integers \mathbb{Z}_p ,

$$\mathbb{Z}_p := \varprojlim_{r \in \mathbb{N}} C_r.$$

The group \mathbb{Z}_p can be regarded as:

$$\mathbb{Z}_p = \{ \mathbf{t}^{\underline{a}} := (t_1^{a_1}, t_2^{a_2}, \dots) \in \prod C_i ; a_r \in \mathbb{Z}, a_r \equiv a_s \pmod{p^s} \text{ whenever } r \geq s \}.$$

Notice that the multiplication in \mathbb{Z}_p is then defined as:

$$\mathbf{t}^{\underline{a}} \mathbf{t}^{\underline{b}} = \mathbf{t}^{\underline{a+b}} = (t_1^{a_1+b_1}, t_2^{a_2+b_2}, \dots)$$

In \mathbb{Z}_p the ‘integers’ are the sequences of the form $(a_1, \dots, a_{i-1}, a_i, a_i, \dots) = (a_i, a_i, \dots)$, which after some point are constant. The element $\mathbf{t} := (t_1, t_2, \dots) \in \mathbb{Z}_p$ corresponds to $(1, 1, \dots)$ in the additive notation, so it generates in \mathbb{Z}_p a copy of \mathbb{Z} and we can write $\mathbb{Z} = \langle \mathbf{t} \rangle$.

We shall introduce now the notion of reduced form.

Definition 1. An element $\underline{a} = (a_1, a_2, \dots) \in \mathbb{Z}_p$ is said to be in *reduced form* if each entry $a_r \in \mathbb{Z}/p^r\mathbb{Z}$ is expressed in its unique p -adic expansion:

$$a_r = k_0 + k_1p + k_2p^2 + \dots + k_{r-1}p^{r-1} + p^r\mathbb{Z}$$

where $k_0, \dots, k_{r-1} \in \{0, 1, \dots, p-1\}$. In the multiplicative notation it means that the exponents are in the above reduced form.

1.4. p -adic framed braids. Consider now the group C_r^n . Let us define $t_{r,i} \in C_r^n$ as,

$$t_{r,i} := (1, \dots, 1, t_r, 1, \dots, 1)$$

with t_r in the i th position. Then we have

$$C_r^n = \langle t_{r,1}, t_{r,2}, \dots, t_{r,n} ; t_{r,i}^{p^r} = 1, t_{r,i}t_{r,j} = t_{r,j}t_{r,i} \text{ for all } i, j \rangle$$

(notice that $C_r^n \cong (\mathbb{Z}/p^r\mathbb{Z})^n$.) By componentwise multiplication, the epimorphism (1.7) defines the connecting epimorphism:

$$\begin{array}{ccc} \pi_s^r : C_r^n & \longrightarrow & C_s^n \\ t_{r,i}^m & \mapsto & t_{s,i}^{\vartheta_s^r(m)} \end{array}$$

for all $r \geq s$. Extending to the B_n -part by the identity map gives rise to the connecting epimorphism:

$$(1.9) \quad \begin{array}{ccc} \pi_s^r \cdot \text{id} : \mathcal{F}_{p^r,n} & \longrightarrow & \mathcal{F}_{p^s,n} \\ \sigma_i & \mapsto & \sigma_i \\ t_{r,i}^m & \mapsto & t_{s,i}^{\vartheta_s^r(m)} \end{array}$$

In [4] we defined the p -adic framed braid group on n strands $\mathcal{F}_{p^\infty,n}$ as:

$$\mathcal{F}_{p^\infty,n} := \varprojlim_{r \in \mathbb{N}} \mathcal{F}_{p^r,n} = \varprojlim_{r \in \mathbb{N}} (C_r^n \rtimes B_n)$$

So, a p -adic framed braid is an infinite sequence of modular framed braids with the same braiding part and such that the framings of the i th strand in each element of the sequence give rise to a p -adic integer. View Figure 1. Elements of $\mathcal{F}_{p^\infty,n}$ are denoted β . We recall now from Proposition 4 in [4] that there are group isomorphisms:

$$(1.10) \quad \mathcal{F}_{p^\infty,n} \cong \mathbb{Z}_p^n \rtimes B_n \cong (\varprojlim_{r \in \mathbb{N}} C_r^n) \rtimes B_n$$

The n -tuples of constant sequences form the subgroup $\mathbb{Z}^n = \langle \mathbf{t}_1, \dots, \mathbf{t}_n \rangle$, where $\mathbf{t}_i := (\mathbf{1}, \dots, \mathbf{1}, \mathbf{t}, \mathbf{1}, \dots, \mathbf{1})$ with \mathbf{t} in the i th position. Note that the element $\mathbf{1} := (\mathbf{1}, \dots, \mathbf{1})$ corresponds to the identity framed braid with all framings zero. Of course, $\sigma_i := (\sigma_i, \sigma_i, \dots)$ in the first isomorphism and $\mathbf{t}_i \in \mathbb{Z} \subset \mathbb{Z}_p^n$ gets identified with $(t_{r,i})_r \in \varprojlim_r C_r^n$ in the second isomorphism.

In view of the first isomorphism, a p -adic framed braid splits into the ‘ p -adic framing’ part and the ‘braiding’ part: $\mathbf{t}_1^{\frac{a_1}{p}} \dots \mathbf{t}_n^{\frac{a_n}{p}} \sigma$, that is, to each strand of the braid $\sigma \in B_n$ we attach a p -adic integer (see Figure 1). p -adic framed braids are multiplied by concatenating their braiding parts and collecting the total p -adic framing of each strand to the top:

$$(\mathbf{t}_1^{\frac{a_1}{p}} \dots \mathbf{t}_n^{\frac{a_n}{p}} \sigma)(\mathbf{t}_1^{\frac{b_1}{p}} \dots \mathbf{t}_n^{\frac{b_n}{p}} \tau) := \mathbf{t}_1^{\frac{a_1+b_{\sigma(1)}}{p}} \dots \mathbf{t}_n^{\frac{a_n+b_{\sigma(n)}}{p}} \sigma\tau$$

Finally, isomorphisms (1.10) imply that a p -adic framed braid can be interpreted as a classical braid with framings p -adic integers or as a classical braid, but with infinite cablings replacing each strand, such that the framings of each infinite cable form a p -adic integer. View Figure 1 for the different facets of a p -adic framed braid. In the sequel we shall not distinguish between the isomorphic forms of $\mathcal{F}_{p^\infty,n}$ neither between the different interpretations of corresponding elements in them.

1.5. *The p -adic Yokonuma–Hecke algebra.* For all $r \geq s$ the linear extension of the map (1.9) yields a connecting algebra epimorphism:

$$(1.11) \quad \varphi_s^r : \mathbb{C}\mathcal{F}_{p^r,n} \longrightarrow \mathbb{C}\mathcal{F}_{p^s,n}$$

Now, passing to the quotient to the algebras, we obtain the following connecting algebra epimorphism

$$\phi_s^r : Y_{p^r,n}(u) \longrightarrow Y_{p^s,n}(u)$$

(cf. [4] for details of the construction of ϕ_s^r). So we obtain the inverse system $(Y_{p^r,n}(u), \phi_s^r)$. In [4] we defined the p -adic Yokonuma–Hecke algebra $Y_{p^\infty,n}(u)$ as:

$$Y_{p^\infty,n}(u) := \varprojlim_{r \in \mathbb{N}} Y_{p^r,n}(u).$$

Definition 2. We shall say that an element in $\mathcal{F}_{p^r,n}$ is in (its unique) *reduced form* if its modular framings are reduced in the sense described in Definition 1. Then, by the linear extension, an element in $\mathbb{C}C_r^n$ or in $\mathbb{C}\mathcal{F}_{p^r,n}$ has a (unique) *reduced form*. Further, an element $y + I_{p^r,n}$ in $Y_{p^r,n}(u)$ is in *reduced form* if the element $y \in \mathbb{C}\mathcal{F}_{p^r,n}$ is written in its (unique) reduced form.

Definition 3. An element $\underline{\beta} = (\beta_1, \beta_2, \dots) \in \mathcal{F}_{p^\infty,n}$ is said to be in its (unique) *reduced form* if every entry $\beta_r \in \mathcal{F}_{p^r,n}$ is reduced. In view of the first isomorphism in (1.10), we may also say that $\underline{\beta} = \underline{\mathbf{t}}_1^{a_1} \dots \underline{\mathbf{t}}_n^{a_n} \sigma$ is in reduced form if its p -adic framings $\underline{a}_1, \dots, \underline{a}_n$ are reduced in the sense of Definition 1. Further, by the linear expansion on $\mathcal{F}_{p^\infty,n}$, an element in $\mathbb{C}\mathcal{F}_{p^\infty,n}$ has a (unique) *reduced form*. An element in $\varprojlim_r \mathbb{C}C_r^n$, in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$ or in $Y_{p^\infty,n}(u)$ is said to be in *reduced form* if every entry is reduced in $\mathbb{C}C_r^n$, in $\mathbb{C}\mathcal{F}_{p^r,n}$ or in $Y_{p^r,n}(u)$ respectively.

Note that, by construction: $\varprojlim_r \mathbb{C}C_r^n \subset \varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$ and $\varprojlim_r \mathbb{C}C_r^n \subset Y_{p^\infty,n}(u)$.

Remark 2. In $\mathcal{F}_{p^\infty,n}$ as well as in $\mathbb{C}\mathcal{F}_{p^\infty,n}$, in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$ and in $Y_{p^\infty,n}(u)$ there are no modular relations.

1.6. *The elements $e_{p^\infty,i}$.* We define now the elements

$$(1.12) \quad e_{p^\infty,i} := (e_{p,i}, e_{p^2,i}, \dots)$$

where

$$(1.13) \quad e_{p^r,i} = \frac{1}{p^r} \sum_{m=0}^{p^r-1} t_{r,i}^m t_{r,i+1}^{-m} \in \mathbb{C}C_r^n \subset \mathbb{C}\mathcal{F}_{p^r,n}$$

Lemma 2. $e_{p^\infty,i} \in \varprojlim_r \mathbb{C}C_r^n \subset \varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$ and $e_{p^\infty,i} \in Y_{p^\infty,n}(u)$ for $i = 1, \dots, n-1$.

Proof. We shall show the coherency of the terms in $e_{p^\infty, i}$, that is $\varphi_s^r(e_{p^r, i}) = e_{p^s, i}$ ($r \geq s$). Note first that by the maps (1.11) and (1.9):

$$\varphi_s^r(t_{r, i}^m t_{r, i+1}^{-m}) = \pi_s^r(t_{r, i}^m t_{r, i+1}^{-m}) = t_{s, i}^m t_{s, i+1}^{-m}$$

But, by (1.13), $e_{p^r, i}$ has p^r terms with linear coefficients $\frac{1}{p^r}$ and $e_{p^s, i}$ has p^s terms with linear coefficients $\frac{1}{p^s}$. Yet,

$$e_{p^r, i} = \frac{1}{p^{r-s}} \left(\sum_{m=0}^{p^s-1} \frac{1}{p^s} t_{r, i}^m t_{r, i+1}^{-m} + \sum_{m=p^s}^{2p^s-1} \frac{1}{p^s} t_{r, i}^m t_{r, i+1}^{-m} + \cdots + \sum_{m=p^{r-s}-p^s}^{p^r-1} \frac{1}{p^s} t_{r, i}^m t_{r, i+1}^{-m} \right).$$

The element in $\mathbb{C}C_r^n$ in each of the p^{r-s} sums is equal to $e_{p^s, i}$ in $\mathbb{C}C_s^n$, so $\varphi_s^r(e_{p^r, i}) = e_{p^s, i}$. Moreover, from the first part and from the definition of the Yokonuma–Hecke algebra it follows immediately that $\phi_s^r(e_{p^r, i}) = e_{p^s, i}$. Thus $e_{p^\infty, i} \in Y_{p^\infty, n}(u)$. \square

The elements $e_{p^\infty, i}$ are no more averaged sums but they are still idempotents. Further, setting by construction $g_i := (g_i, g_i, \dots)$ and $1 := (1, 1, \dots)$ we have in $Y_{p^\infty, n}(u)$ the braid relations for the g_i 's and the relations:

$$(1.14) \quad g_i^2 = 1 + (u-1)e_{p^\infty, i} - (u-1)e_{p^\infty, i} g_i$$

and

$$g_i^{-1} = g_i - (u^{-1} - 1)e_{p^\infty, i} + (u^{-1} - 1)e_{p^\infty, i} g_i$$

Moreover, for powers of g_i relations analogous to the ones in Lemma 1 are valid in $Y_{p^\infty, n}(u)$, after replacing $e_{d, i}$ by $e_{p^\infty, i}$. Finally, using the elements $e_{d, i, k}$ in (1.6), we can define for $i = 1, \dots, n-1$ and $k \neq i$ the elements:

$$(1.15) \quad e_{p^\infty, i, k} := (e_{p, i, k}, e_{p^2, i, k}, \dots) \in \varprojlim_{r \in \mathbb{N}} \mathbb{C}C_r^n \subset \varprojlim_{r \in \mathbb{N}} \mathbb{C}\mathcal{F}_{p^r, n}$$

abbreviating $e_{p^\infty, i, i+1}$ to $e_{p^\infty, i}$. Clearly, $e_{p^\infty, i, k} = e_{p^\infty, k, i}$ and $e_{p^\infty, i, k}^2 = e_{p^\infty, i, k}$. These elements satisfy the following relations, cf. Lemma 7 in [4] and Proposition 10 in [4]:

$$(1.16) \quad \begin{aligned} e_{p^\infty, j} e_{p^\infty, i} &= e_{p^\infty, i} e_{p^\infty, j} \\ g_j e_{p^\infty, i} &= e_{p^\infty, i} g_j & \text{for } j \neq i-1, i+1 \\ g_{i-1} e_{p^\infty, i} &= e_{p^\infty, i-1, i+1} g_{i-1} & \text{and } e_{p^\infty, i} g_{i-1} = g_{i-1} e_{i-1, i+1} \\ g_{i+1} e_{p^\infty, i} &= e_{p^\infty, i, i+2} g_{i+1} & \text{and } e_{p^\infty, i} g_{i+1} = g_{i+1} e_{p^\infty, i, i+2} \end{aligned}$$

Relations (1.16) are also valid if all g_k 's are replaced by their inverses g_k^{-1} .

1.7. Comparing algebras. It is worth stressing at this point that, despite the definition $\mathcal{F}_{p^\infty, n} = \varprojlim_r \mathcal{F}_{p^r, n}$, the algebras $\mathbb{C}\mathcal{F}_{p^\infty, n}$ and $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ are not isomorphic. Before stating our result let us take a closer look at the two algebras. $\mathbb{C}\mathcal{F}_{p^\infty, n}$ consists in all finite linear expressions of the form

$$\lambda_1 \underline{b}_1 + \dots + \lambda_k \underline{b}_k$$

where $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $\underline{b}_1, \dots, \underline{b}_k \in \mathcal{F}_{p^\infty, n}$. On the other hand, elements in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ are coherent sequences of elements in the group algebras $\mathbb{C}\mathcal{F}_{p^r, n}$ in the sense of the map (1.11). That is, given elements $\beta_1, \dots, \beta_k \in \mathcal{F}_{p^r, n}$ and $c_1, \dots, c_k \in \mathbb{C}$ then

$$\varphi_s^r(c_1\beta_1 + \dots + c_k\beta_k) = c_1(\pi_s^r \cdot \text{id})(\beta_1) + \dots + c_k(\pi_s^r \cdot \text{id})(\beta_k) \in \mathbb{C}\mathcal{F}_{p^s, n}$$

By construction of the map φ_s^r it appears as though all positions of a p -adic element in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ have the same number of coherent monomials with the same coefficients. This form of a p -adic element is always possible by construction, but it may be hidden, since the linear coefficients may allow for various manipulations. To appreciate this subtlety in the form of elements in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ consider, as typical example, the elements $e_{p^\infty, i}$ defined in (1.12).

On the level of p -adic braids, $\mathcal{F}_{p^\infty, n}$ can be regarded naturally in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$. So, we obtain a natural linear map

$$f : \mathbb{C}\mathcal{F}_{p^\infty, n} \longrightarrow \varprojlim_{r \in \mathbb{N}} \mathbb{C}\mathcal{F}_{p^r, n}$$

Let us see how exactly the map f works. Consider $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $\underline{b}_1, \dots, \underline{b}_k$ different elements in $\mathcal{F}_{p^\infty, n}$, where $\underline{b}_i = (b_{ri})_r$ with $b_{ri} \in \mathcal{F}_{p^r, n}$. Then

$$\begin{aligned} & \lambda_1 \underline{b}_1 + \dots + \lambda_k \underline{b}_k \in \mathbb{C}\mathcal{F}_{p^\infty, n} \\ & \parallel \\ & \lambda_1(b_{11}, b_{21}, \dots) + \dots + \lambda_k(b_{1k}, b_{2k}, \dots) \\ & \downarrow f \\ & (\lambda_1 b_{11}, \lambda_1 b_{21}, \dots) + \dots + (\lambda_k b_{1k}, \lambda_k b_{2k}, \dots) \\ & \parallel \\ & (\lambda_1 b_{11} + \dots + \lambda_k b_{1k}, \lambda_1 b_{21} + \dots + \lambda_k b_{2k}, \dots) \in \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n} \end{aligned}$$

By construction, f is constant on $\mathcal{F}_{p^\infty, n}$, so $f(\underline{b}) = \underline{b}$. Also, by linearity, the image of f is generated by all $f(\underline{b})$ where $\underline{b} \in \mathcal{F}_{p^\infty, n}$.

Lemma 3. $e_{p^\infty, i} \notin f(\mathbb{C}\mathcal{F}_{p^\infty, n})$.

Proof. Suppose that $e_{p^\infty, i} \in f(\mathbb{C}\mathcal{F}_{p^\infty, n})$. Then, from the above, $e_{p^\infty, i} = a_1 \underline{b}_1 + \dots + a_k \underline{b}_k$ for some $a_1, \dots, a_k \in \mathbb{C}$ and $\underline{b}_1, \dots, \underline{b}_k$ as above. Then, by the structure of $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ as linear space, we have in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ the equality:

$$(e_{p, i}, e_{p^2, i}, \dots) = (a_1 b_{11} + \dots + a_k b_{1k}, a_1 b_{21} + \dots + a_k b_{2k}, \dots)$$

Equivalently, in each $\mathbb{C}\mathcal{F}_{p^r, n}$, $r = 1, 2, \dots$ we have the equality:

$$(1.17) \quad \sum_{m=1}^{p^r} t_{r,i}^m t_{r,i+1}^{-m} = \sum_{j=1}^k p^r a_j b_{rj}$$

Since $\underline{b}_1, \dots, \underline{b}_k$ are different in $\mathcal{F}_{p^\infty, n}$ there must exist some $s \in \mathbb{N}$ such that b_{s1}, \dots, b_{sk} are different elements in $\mathcal{F}_{p^s, n}$ and this is then true for any $r \geq s$. So, there exists some $r \geq s$ such that $k < p^r$. But then (1.17) states equality of two linear expressions of linearly independent elements in $\mathbb{C}\mathcal{F}_{p^r, n}$. Since $k < p^r$, all coefficients $p^r a_j$ must be equal to 1. Subtracting we obtain a summation of terms $t_{r,i}^m t_{r,i+1}^{-m}$ equal to zero, contradiction since they are linearly independent. Therefore $e_{p^\infty, i} \notin f(\mathbb{C}\mathcal{F}_{p^\infty, n})$. \square

Corollary 1. $e_{p^\infty, i} \notin \mathbb{C}\mathcal{F}_{p^\infty, n}$.

Proof. Suppose $e_{p^\infty, i} \in \mathbb{C}\mathcal{F}_{p^\infty, n}$. Then $e_{p^\infty, i} = a_1 \underline{b}_1 + \dots + a_k \underline{b}_k$ for some $a_1, \dots, a_k \in \mathbb{C}$ and $\underline{b}_1, \dots, \underline{b}_k$ as above. But then $f(e_{p^\infty, i}) = e_{p^\infty, i}$, not possible by Lemma 3. Hence $e_{p^\infty, i} \notin \mathbb{C}\mathcal{F}_{p^\infty, n}$. \square

Proposition 1. *The linear map f is injective but not surjective. Hence, $\mathbb{C}\mathcal{F}_{p^\infty, n} \cong f(\mathbb{C}\mathcal{F}_{p^\infty, n})$ and so the algebra $\mathbb{C}\mathcal{F}_{p^\infty, n}$ can be regarded as a proper subalgebra of $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$.*

Proof. We will show that f is injective. With the above notations let

$$f(\lambda_1 \underline{b}_1 + \dots + \lambda_k \underline{b}_k) = (0, 0, \dots)$$

Equivalently, $\lambda_1 b_{r1} + \dots + \lambda_k b_{rk} = 0$ in $\mathbb{C}\mathcal{F}_{p^r, n}$ for all $r = 1, 2, \dots$. As in the proof of Lemma 3, since $\underline{b}_1, \dots, \underline{b}_k$ are different in $\mathcal{F}_{p^\infty, n}$ there must exist some $r \in \mathbb{N}$ such that b_{r1}, \dots, b_{rk} are different elements in $\mathcal{F}_{p^r, n}$. Hence they are linearly independent in $\mathbb{C}\mathcal{F}_{p^r, n}$, hence $\lambda_1 = \dots = \lambda_k = 0$. Therefore $\text{Ker } f = \{0\}$ and so f is injective.

The fact that f is not surjective follows immediately from Lemmas 2 and 3, since $e_{p^\infty, i} \in \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ but $e_{p^\infty, i} \notin f(\mathbb{C}\mathcal{F}_{p^\infty, n})$. \square

2. DENSE SUBSETS AND APPROXIMATIONS OF p -ADIC ELEMENTS

2.1. A general lemma. Our method for finding dense subsets in our p -adic structures is by means of the following known result.

Lemma 4. (cf. [8], Lemma 1.1.7.) *Let ρ_i denote the restriction of the canonical projection of $\varprojlim X_i$ onto X_i on a subset $A \subset \varprojlim X_i$. Recall that $\varprojlim A$ can be identified with A . If $\rho_i(A) = X_i$ for all $i \in I$, then $\rho(\varprojlim A)$ is dense in $\varprojlim X_i$, where $\rho = \varprojlim \rho_i : \varprojlim A \longrightarrow \varprojlim X_i$, the induced mapping.*

Definition 4. (cf. [8] § 2.4) Let G_i be a group (ring, algebra, et cetera) for all $i \in I$. A subset $S \subset \varprojlim G_i$ is a set of *topological generators* of $\varprojlim G_i$ if the span $\langle S \rangle$ is dense in $\varprojlim G_i$. If, moreover, S is finite, $\varprojlim G_i$ is said to be *finitely generated*.

Our method for finding approximating sequences of p -adic elements is by strict inclusions of open neighborhoods. As a topological space, $\prod X_i$ is endowed with the product topology, so $\varprojlim X_i$ inherits the induced topology. It

can be then verified that $\varprojlim X_i$ is closed in $\prod X_i$. A *basis of open sets* in $\varprojlim X_i$ contains elements of the form

$$\varpi_i^{-1}(U_i) \cap \varprojlim X_i$$

where U_i open in X_i . Then, any open set in $\varprojlim X_i$ is a union of sets of the form

$$\varpi_{i_1}^{-1}(U_1) \cap \dots \cap \varpi_{i_n}^{-1}(U_n) \cap \varprojlim X_i$$

where $i_1, \dots, i_n \in I$ and U_r open in X_{i_r} for each r . (Compare with [8], p.7.)

2.2. Approximations in \mathbb{Z}_p . Since \mathbb{Z} projects onto each factor $\mathbb{Z}/p^r\mathbb{Z}$, by Lemma 4, the image of \mathbb{Z} under the induced map on the inverse limits is dense in \mathbb{Z}_p . Now $\varprojlim_r \mathbb{Z} = \mathbb{Z}$ and the induced map acts on an element $(x, x, \dots) \in \mathbb{Z}$ by sending x to $x \pmod{p^r} \in \mathbb{Z}/p^r\mathbb{Z}$ for every r . But, after some point x will be unchanged by the modulus, so $(x \pmod{p}, x \pmod{p^2}, \dots) = (x, x, \dots)$. Therefore, the image of \mathbb{Z} under the induced map on the inverse limits is \mathbb{Z} , and \mathbb{Z} is dense in \mathbb{Z}_p .

Now $\mathbb{Z} = \langle \mathbf{t} \rangle$, so \mathbf{t} is a topological generator of \mathbb{Z}_p . Thus, an element $\mathbf{t}^a = (t_1^{a_1}, t_2^{a_2}, \dots)$ in \mathbb{Z}_p is approximated by constant sequences, which are identified with integers. We shall explain how to find such an approximating sequence for a p -adic integer, in order to draw the strategy for the larger p -adic structures we are dealing with.

The inherited topology of \mathbb{Z}_p builds up from the discrete topology of each factor $\mathbb{Z}/p^r\mathbb{Z}$. Thus, a basic open set U in \mathbb{Z}_p is of the form $U = \varpi_i^{-1}(U_i)$; $U_i \subseteq \mathbb{Z}/p^i\mathbb{Z}$. For U_i not a singleton, $\varpi_i^{-1}(U_i) = \cup_{u \in U_i} \varpi_i^{-1}(\{u\})$.

Recall now Definition 1. It is then easy to verify the lemma below.

Lemma 5. *Let $\underline{a} = (a_1, a_2, a_3, \dots) \in \mathbb{Z}_p$ in reduced form and let $U_i \subseteq \mathbb{Z}/p^i\mathbb{Z}$. Then $\underline{a} \in U = \varpi_i^{-1}(U_i)$ if and only if $a_i \in U_i$. Hence, a basic open neighborhood of \underline{a} in \mathbb{Z}_p is of the form $U = \varpi_i^{-1}(\{a_i\})$ for some i . Moreover, we have a nested sequence of neighborhoods with strict inclusions:*

$$\varpi_1^{-1}(\{a_1\}) \supsetneq \varpi_2^{-1}(\{a_2\}) \supsetneq \dots$$

By the strict inclusions of neighborhoods, the sequence of constant sequences $((a_k))_{k \in \mathbb{N}}$ in \mathbb{Z} approximates $\underline{a} \in \mathbb{Z}_p$ and we write $\underline{a} = \lim_k (a_k)$ or, in the multiplicative notation:

$$\mathbf{t}^a = \lim_k \mathbf{t}^{a_k}$$

Indeed, subtracting each constant sequence successively from \underline{a} the differences tend to the zero sequence:

$$\begin{aligned} (a_1, a_2, a_3, \dots) - (a_1, a_1, a_1, \dots) &= (0, a_2 - a_1, a_3 - a_1, \dots) \\ (a_1, a_2, a_3, \dots) - (a_2, a_2, a_2, \dots) &= (0, 0, a_3 - a_2, \dots) \\ &\vdots \end{aligned}$$

In this sense the element \underline{a} can be viewed as the infinite power series:

$$\underline{a} = \sum_{r=0}^{\infty} k_r p^r \text{ where } k_r \in \{0, 1, \dots, p-1\}$$

In order to draw a general scheme for finding approximating sequences to p -adic elements we shall introduce the operations truncation and expansion for entries of p -adic integers. Let $\underline{a} = (a_1, a_2, \dots) \in \mathbb{Z}_p$ in reduced form. For any indices r, s with $r > s$ we define the s -truncation of a_r as the element

$$a_{rs} = k_0 + k_1 p + \dots + k_{s-1} p^{s-1} + p^s \mathbb{Z} \in \mathbb{Z}/p^s \mathbb{Z}$$

Note that $\theta_s^r(a_{rs}) = a_s$, that is, the elements a_{rs} and a_s are coherent via the map 1.8.

Similarly, for $r < s$, we define the r -expansion of a_s as the element

$$a_{sr} = k_0 + k_1 p + \dots + k_{r-1} p^{r-1} + p^r \mathbb{Z} \in \mathbb{Z}/p^r \mathbb{Z}$$

Note that $a_{sr} \equiv a_s \pmod{p^s}$, so $\theta_s^r(a_r) = a_{sr}$. In the multiplicative notation $t_r^{a_r}$ is substituted by $t_r^{a_s}$ in the first case and $t_s^{a_s}$ is substituted by $t_s^{a_r}$ in the second case. The fact that \underline{a} is in reduced form ensures that truncations and expansions of its entries are well-defined.

In this terminology, the constant sequences (a_k) approximating \underline{a} are found from \underline{a} by truncating each term after a_k with respect to a_k and expanding each term before a_k with respect to a_k .

2.3. Approximations in $\mathcal{F}_{p^\infty, n}$ and $\mathbb{C}\mathcal{F}_{p^\infty, n}$. Applying the canonical epimorphism (1.7) componentwise yields a canonical epimorphism of \mathbb{Z}^n on each factor C_r^n . So, by Lemma 4, \mathbb{Z}^n is dense in \mathbb{Z}_p^n . Then, for example, for $\underline{a} = (a_1, a_2, \dots)$ and $\underline{b} = (b_1, b_2, \dots) \in \mathbb{Z}_p$ in reduced form, the element $(\underline{a}, \underline{b}) \in \mathbb{Z}_p^2$ is approximated by the sequence $((a_k, b_k))_{k \in \mathbb{N}}$ of constant sequences, that is, with terms in the dense subgroup \mathbb{Z}^2 . In multiplicative notation: $(\mathbf{t}_1^{\underline{a}}, \mathbf{t}_2^{\underline{b}}) = \mathbf{t}_1^{\underline{a}} \mathbf{t}_2^{\underline{b}} \in \mathbb{Z}_p^2$ is approximated by the sequence $((\mathbf{t}_1^{a_k}, \mathbf{t}_2^{b_k}))_{k \in \mathbb{N}} = ((\mathbf{t}_1^{a_k} \mathbf{t}_2^{b_k}))_{k \in \mathbb{N}}$ with terms in \mathbb{Z}^2 .

Further, we extend the projection of \mathbb{Z}^n on each factor C_r^n by the identity map on B_n . So, we obtain an epimorphism of the classical framed braid group $\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$ on each factor $\mathcal{F}_{p^r, n}$. Hence, by Lemma 4, \mathcal{F}_n is dense in $\mathcal{F}_{p^\infty, n}$. The set $\{\mathbf{t}_1, \dots, \mathbf{t}_1, \sigma_1, \dots, \sigma_{n-1}\}$ is a set of topological generators for $\mathcal{F}_{p^\infty, n}$ satisfying relations similar to the relations of \mathcal{F}_n . Moreover, for $\underline{a_i} = (a_{ri})_r$ in reduced form, an element $\underline{\beta} = \mathbf{t}_1^{a_1} \dots \mathbf{t}_n^{a_n} \sigma \in \mathcal{F}_{p^\infty, n}$ has the approximation:

$$\underline{\beta} = \lim_k (\mathbf{t}_1^{a_{k1}} \dots \mathbf{t}_n^{a_{kn}} \sigma)$$

An example is illustrated in Figure 5.

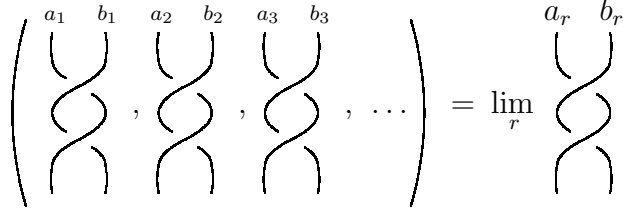


FIGURE 5. Approximating a p -adic braid by classical braids

Passing to algebras, for an element in $\mathbb{C}\mathcal{F}_{p^\infty, n}$ in reduced form (recall Definition 3) it is easy to find an approximating sequence with elements in the dense algebra $\mathbb{C}\mathcal{F}_n$. Indeed, we simply extend linearly the approximations of its monomials in $\mathcal{F}_{p^\infty, n}$, as described above. Note that, by construction, the p -adic element may always be written in the form where the linear combinations in each place have the same number of coherent monomials with the same coefficients.

We would like now to find approximating sequences for elements in $\varprojlim_r \mathbb{C}C_r^n$, in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ and in $Y_{p^\infty, n}(u)$. The tactics used for $\mathbb{C}\mathcal{F}_{p^\infty, n}$, that is approximating each monomial, cannot be applied here for purely p -adic elements (such as $e_{p^\infty, i}$) since they cannot be written in the form where the linear combinations in each place have the same number of coherent terms with the same coefficients. So, finding an approximating sequence for purely p -adic elements is more tricky. In any case, we need first to find dense subalgebras of constant elements, in which the approximating terms should live.

2.4. Dense subsets in the p -adic algebras. Extending linearly the epimorphism of \mathcal{F}_n on each factor $\mathcal{F}_{p^r, n}$ defines an epimorphism η_r of the algebra $\mathbb{C}\mathcal{F}_n$ on the algebra $\mathbb{C}\mathcal{F}_{p^r, n}$. Moreover, the map η_r composed with the canonical epimorphism ρ_r defines an epimorphism μ_r of $\mathbb{C}\mathcal{F}_n$ on the algebra $Y_{p^r, n}(u)$:

$$(2.1) \quad \begin{array}{ccccc} \mathbb{C}\mathcal{F}_n & \xrightarrow{\eta_r} & \mathbb{C}\mathcal{F}_{p^r, n} & \xrightarrow{\rho_r} & Y_{p^r, n}(u) \\ \sigma_i & \mapsto & \sigma_i & \mapsto & g_i \\ \mathbf{t}_j^m & \mapsto & t_{r, j}^{\vartheta_s^r(m)} & \mapsto & t_{r, j}^{\vartheta_s^r(m)} \end{array}$$

Define further $\eta := \varprojlim_r \eta_r$, $\rho := \varprojlim_r \rho_r$ and $\mu := \varprojlim_r \mu_r$, the corresponding induced maps on the inverse limits of the maps (2.1):

$$(2.2) \quad \eta : \mathbb{C}\mathcal{F}_n \longrightarrow \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$$

$$(2.3) \quad \rho : \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n} \longrightarrow Y_{p^\infty, n}(u)$$

and

$$(2.4) \quad \mu : \mathbb{C}\mathcal{F}_n \longrightarrow Y_{p^\infty, n}(u)$$

Recall that $\varprojlim_r \mathbb{C}\mathcal{F}_n = \mathbb{C}\mathcal{F}_n$. One can easily check that $\eta(\sigma_i) = \sigma_i$ and $\eta(\mathbf{t}_j) = \mathbf{t}_j$, so η is an injection. Also, that $\mu = \rho \circ \eta$. Moreover, by the construction of the maps (2.1) and (2.4) we have that $\mu(\sigma_i) = g_i$ and $\mu(\mathbf{t}_j) = \mathbf{t}_j$. Below, in Proposition 2, we also show that ρ is a surjection.

From (2.1)–(2.4) we have the following theorem.

Theorem 1. (1) *The algebra $\mathbb{C}\mathcal{F}_n$ is dense in $\mathbb{C}\mathcal{F}_{p^\infty, n}$.*
 (2) *The algebra $\mathbb{C}\mathcal{F}_n$ is dense in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$.*
 (3) *The set $X = \{\mathbf{1}, \mathbf{t}_1, \dots, \mathbf{t}_n, \sigma_1, \dots, \sigma_{n-1}\}$ is a set of topological generators for the algebras $\mathbb{C}\mathcal{F}_{p^\infty, n}$ and $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$. Together with the braid relations for the σ_i 's, the commuting relations for the \mathbf{t}_j 's and the relations:*

$$\sigma_i \mathbf{t}_i = \mathbf{t}_{i+1} \sigma_i, \sigma_i \mathbf{t}_{i+1} = \mathbf{t}_i \sigma_i \text{ and } \sigma_i \mathbf{t}_j = \mathbf{t}_j \sigma_i \text{ for } j \neq i, i+1$$

they furnish a topological presentation for $\mathbb{C}\mathcal{F}_{p^\infty, n}$ and $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$.

(4) *The algebra $\mu(\mathbb{C}\mathcal{F}_n)$ is dense in $Y_{p^\infty, n}(u)$. Moreover, the set:*

$$D = \{\mathbf{1}, \mathbf{t}_1, \dots, \mathbf{t}_n, g_1, \dots, g_{n-1}\}$$

is a set of topological generators for the algebra $Y_{p^\infty, n}(u)$, satisfying the analogous relations of (iii) in $\mu(\mathbb{C}\mathcal{F}_n)$.

(5) *The relations $\mathbf{t}_j e_i = e_i \mathbf{t}_j$ are valid in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ and in $Y_{p^\infty, n}(u)$, but not in the dense subalgebras. Similarly for the quadratic relations (1.14) in $Y_{p^\infty, n}(u)$.*

Proof. Since \mathcal{F}_n is dense in $\mathcal{F}_{p^\infty, n}$, claim (i) follows by linear extension. Claim (ii) is an application of the surjections (2.1), Lemma 4 and of the following observation: after some point the image of the exponent m in (2.1) will not change in each position, so $\eta(\mathbb{C}\mathcal{F}_n) = \mathbb{C}\mathcal{F}_n$, where η the map (2.2).

It follows now from (ii) that the set X is a set of topological generators, satisfying the listed relations, cf. also Theorem 3[4]. Moreover, by the standard presentation of the classical framed braid group \mathcal{F}_n (recall (1.1)), the relations given in claim (iii) are the only ones satisfied in the set X .

The fact that $\mu(\mathbb{C}\mathcal{F}_n)$ is dense in $Y_{p^\infty, n}(u)$ is clear by a direct application of the surjections (2.1) and Lemma 4. Further, $\mu(\mathbf{t}_j) = \mathbf{t}_j$ and $\mu(\sigma_i) = g_i$. So, claim (iv) follows.

Finally, claim (v) follows from claims (iii) and (iv), from Corollary 1 and from the fact that $\mu(e_{p^\infty, i}) = e_{p^\infty, i}$. \square

Focusing now a little more on $Y_{p^\infty, n}(u)$, an apparently dense subset in $Y_{p^\infty, n}(u)$, discussed in [4], comes from the following construction. For any r we have the following exact sequence:

$$(2.5) \quad 0 \longrightarrow I_{p^r, n} \xrightarrow{\iota_r} \mathbb{C}\mathcal{F}_{p^r, n} \xrightarrow{\rho_r} Y_{p^r, n}(u) \longrightarrow 0$$

where $I_{p^r,n}$ is the ideal generated by the linear expression related to (1.4). This induces the exact sequence:

$$0 \longrightarrow \varprojlim_r I_{p^r,n} \xrightarrow{\iota} \varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n} \xrightarrow{\rho} Y_{p^\infty,n}(u)$$

where $\iota := \varprojlim_r \iota_r$ and $\rho := \varprojlim_r \rho_r$. Hence, and since $\varprojlim_r I_{p^r,n}$ is an ideal in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$, we have:

$$\frac{\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}}{\varprojlim_r I_{p^r,n}} \cong \rho(\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n})$$

At the writing of [4] it was not clear whether the map ρ is a surjection or not. Yet, by application of Lemma 4 we could derive the result that $\rho(\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n})$ is dense in $Y_{p^\infty,n}(u)$.

We are now in a position to prove surjection for ρ . Before that we need to recall the following definition. An inverse system $X = (X_i, \phi_j^i)$ indexed by \mathbb{N} is said to satisfy the *ML-condition* (Mittag-Leffler condition) if for any index m there exists $n \geq m$ such that for all $n' \geq n$ we have $\text{Im}(\phi_m^n) = \text{Im}(\phi_m^{n'})$. Notice that if all ϕ_j^i are surjective then X satisfies the ML-condition.

Proposition 2. $\rho(\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}) = Y_{p^\infty,n}(u)$. Hence $Y_{p^\infty,n}(u) \cong \frac{\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}}{\varprojlim_r I_{p^r,n}}$.

Proof. The exact sequence (2.5) induces the following exact sequence of inverse systems.

$$0 \longrightarrow (I_{p^r,n}, \varphi_s^r) \xrightarrow{(\iota_r)} (\mathbb{C}\mathcal{F}_{p^r,n}, \varphi_s^r) \xrightarrow{(\rho_r)} (Y_{p^r,n}, \phi_s^r) \longrightarrow 0$$

Now, by Lemma 6[4], $\varphi_s^r(I_{p^r,n}) = I_{p^s,n}$, hence the inverse system $(I_{p^r,n}, \varphi_s^r)$ satisfies the ML-condition. Then, by a well-known result of Grothendieck the following exact sequence is induced:

$$0 \longrightarrow \varprojlim_r I_{p^r,n} \xrightarrow{\iota} \varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n} \xrightarrow{\rho} Y_{p^\infty,n}(u) \longrightarrow 0$$

Hence ρ is surjection. \square

We shall now recapitulate what we know and what we don't know about our p -adic objects, by means of a concise diagram. For that we need to introduce two more intermediate structures.

Definition 5. We define the dense subalgebra $\widetilde{\mathbb{C}\mathcal{F}_n}$ of $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$ as the extension of the subalgebra $\mathbb{C}\mathcal{F}_n$ by the elements e_1, \dots, e_{n-1} . We also define the dense subalgebra $\widetilde{Y}_n(u)$ of $Y_{p^\infty,n}(u)$ as the extension of $Y_n(u) := \mu(\mathbb{C}\mathcal{F}_n)$ by the elements e_1, \dots, e_{n-1} .

Clearly $\widetilde{\mathbb{C}\mathcal{F}_n}$ is a proper subset of $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r,n}$. For example, $\widetilde{\mathbb{C}\mathcal{F}_n}$ does not contain the p -adic integers. By the same reason $\widetilde{Y}_n(u)$ is also a strict subset of $Y_{p^\infty,n}(u)$. Moreover, denoting $\tilde{\mu}$ the restriction of ρ on $\widetilde{\mathbb{C}\mathcal{F}_n}$ we have that

$\widetilde{\mu}(\widetilde{\mathbb{CF}}_n) = \widetilde{Y}_n(u)$, since $\widetilde{\mu}(e_{p^\infty,i}) = e_{p^\infty,i}$. Further, the quadratic relations (1.14) are valid in $\widetilde{Y}_n(u)$ and, by construction, $\widetilde{Y}_n(u)$ is the smallest subalgebra of $Y_{p^\infty,n}(u)$ which is closed under the quadratic relations (1.14).

The relations in claim (iv) of Theorem 1 together with the quadratic relations (1.14) form then a complete set of relations for $\widetilde{Y}_n(u)$. Thus, $\widetilde{Y}_n(u)$ can be viewed as the quotient:

$$\widetilde{Y}_n(u) = \frac{\widetilde{\mathbb{CF}}_n}{\langle g_i^2 - 1 - (u-1)e_{p^\infty,i} + (u-1)e_{p^\infty,i}g_i \rangle}$$

and so, $Y_{p^\infty,n}(u)$ can be regarded as a topological deformation of the above quotient algebra.

$$\begin{array}{ccccc} e_{p^\infty,i} \notin \mathbb{CF}_n & \xrightarrow{\eta=\iota} & \widetilde{\mathbb{CF}}_n = \langle \mathbb{CF}_n, e_{p^\infty,i} \rangle & \xrightarrow[\mathcal{Q}]{\iota} & \varprojlim \mathcal{F}_{p^r,n} \\ \downarrow \mu & & \downarrow \widetilde{\mu} & & \downarrow \rho \\ e_{p^\infty,i} \notin \mu(\mathbb{CF}_n) & \xrightarrow{\iota} & \widetilde{Y}_n(u) = \langle \mu(\mathbb{CF}_n), e_{p^\infty,i} \rangle & \xrightarrow[\mathcal{Q}]{\iota} & Y_{p^\infty,n}(u) \end{array}$$

With our dense subalgebras in hand, we shall next discuss an approximation for the elements $e_{p^\infty,i}$ before going to the general case of approximating purely p -adic elements.

2.5. Approximating $e_{p^\infty,i}$. Recall from (1.13) that an entry $e_{p^r,i}$ of the element $e_{p^\infty,i} = (e_{p,i}, e_{p^2,i}, \dots) \in \varprojlim_r \mathbb{CC}_r^n \subset \varprojlim_r \mathbb{CF}_{p^r,n}$ has p^r terms with linear coefficients $\frac{1}{p^r}$ and this is in reduced form, according to Definition 3. We are looking for an approximating sequence for $e_{p^\infty,i}$, consisting of constant terms. Recall from the proof of Lemma 2 that $e_{p^r,i}$ can be arranged in the form:

$$(2.6) \quad e_{p^r,i} = \frac{1}{p^{r-s}} \left(\sum_{m=0}^{p^s-1} \frac{1}{p^s} t_{r,i}^m t_{r,i+1}^{-m} + \sum_{m=p^s}^{2p^s-1} \frac{1}{p^s} t_{r,i}^m t_{r,i+1}^{-m} + \dots + \sum_{m=p^{r-1}-p^s}^{p^r-1} \frac{1}{p^s} t_{r,i}^m t_{r,i+1}^{-m} \right)$$

of p^{r-s} packets, each of which projects on $e_{p^s,i}$ by the coherency map φ_s^r ($r \geq s$). We shall define the first packet as the s -truncation of $e_{p^r,i}$. On the other hand, in order to make the linear expression for $e_{p^s,i}$ agree formally with that for $e_{p^r,i}$, we rewrite $e_{p^s,i}$ as a sum of totally p^r terms, arranged in p^{r-s} packets, each of which is equal to $e_{p^s,i}$:

$$(2.7) \quad e_{p^s,i} = \frac{1}{p^{r-s}} \left(\sum_{m=0}^{p^s-1} \frac{1}{p^s} t_{s,i}^m t_{s,i+1}^{-m} + \sum_{m=p^s}^{2p^s-1} \frac{1}{p^s} t_{s,i}^m t_{s,i+1}^{-m} + \dots + \sum_{m=p^{r-1}-p^s}^{p^r-1} \frac{1}{p^s} t_{s,i}^m t_{s,i+1}^{-m} \right)$$

We shall define this as the r -expansion of $e_{p^s, i}$.

Definition 6. For any indices r, s with $r \geq s$ we define the element $\zeta_{r, s, i}$ in $\mathbb{C}C_r^n \subset \mathbb{C}\mathcal{F}_{p^r, n}$ as the formal expression of $e_{p^s, i}$, but with the generators $t_{s, i}, t_{s, i+1}$ replaced by the generators $t_{r, i}, t_{r, i+1}$. That is:

$$\zeta_{r, s, i} := \frac{1}{p^s} \sum_{m=0}^{p^s-1} t_{r, i}^m t_{r, i+1}^{-m} \in \mathbb{C}\mathcal{F}_{p^r, n}$$

The element $\zeta_{r, s, i}$ is called the s -truncation of $e_{p^r, i}$. Note that $\zeta_{r, r, i} = e_{p^r, i}$. Clearly, the elements $\zeta_{r, s, i}$ and $e_{p^s, i}$ are coherent: $\varphi_s^r(\zeta_{r, s, i}) = e_{p^s, i}$. In fact, $\zeta_{r, s, i}$ ‘wraps’ only once on $e_{p^s, i}$ via the map φ_s^r .

Further, we define the element $\zeta_{s, r, i}$ in $\mathbb{C}C_s^n \subset \mathbb{C}\mathcal{F}_{p^s, n}$ as the formal expression of $e_{p^r, i}$, but with the generators $t_{r, i}, t_{r, i+1}$ replaced by the generators $t_{s, i}, t_{s, i+1}$. That is:

$$\zeta_{s, r, i} := \frac{1}{p^r} \sum_{m=0}^{p^r-1} t_{s, i}^m t_{s, i+1}^{-m} \in \mathbb{C}\mathcal{F}_{p^s, n}$$

The element $\zeta_{s, r, i}$ is called the r -expansion of $e_{p^s, i}$. Clearly, $\zeta_{s, r, i} = e_{p^s, i}$.

We define now the element $\mathbf{e}_{p^r, i}$ by r -truncating each term in $e_{p^\infty, i}$ after the r th position and by r -expanding each term before the r th position. That is,

$$\mathbf{e}_{p^r, i} := (\zeta_{1, r, i}, \zeta_{2, r, i}, \dots, \zeta_{r-1, r, i}, e_{p^r, i}, \zeta_{r+1, r, i}, \zeta_{r+2, r, i}, \dots)$$

Proposition 3. For any index r the element $\mathbf{e}_{p^r, i}$ is a constant sequence in $\varprojlim_r \mathbb{C}C_r^n \subset \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$. More precisely:

$$\mathbf{e}_{p^r, i} = \left(\frac{1}{p^r} \sum_{m=0}^{p^r-1} t_{k, i}^m t_{k, i+1}^{-m} \right)_k = \frac{1}{p^r} \sum_{m=0}^{p^r-1} \mathbf{t}_i^m \mathbf{t}_{i+1}^{-m} \in \mathbb{C}\mathcal{F}_n$$

Moreover, we have the approximation:

$$e_{p^\infty, i} = \lim_r \mathbf{e}_{p^r, i}$$

In the above sense $e_{p^\infty, i}$ can be viewed as an infinite series:

$$e_{p^\infty, i} = \sum_{r=0}^{\infty} \mathbf{e}_{p^r, i} \in \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$$

Proof. By Definition 7 the sequence $\mathbf{e}_{p^r, i}$ is coherent, so $\mathbf{e}_{p^r, i} \in \varprojlim_r \mathbb{C}C_r^n \subset \varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$. Moreover, all terms in $\mathbf{e}_{p^r, i}$ have the same formal expression, that of $e_{p^r, i}$, so $\mathbf{e}_{p^r, i}$ is the constant sequence given in the statement. Finally, recall from (1.10) that $(t_{r, i})_r = \mathbf{t}_i \in \mathbb{Z}_p^n$. So, separating terms in $\mathbf{e}_{p^r, i}$ we obtain:

$$\mathbf{e}_{p^r, i} = \frac{1}{p^r} \sum_{m=0}^{p^r-1} (t_{1, i}^m, t_{2, i}^m, \dots) (t_{1, i+1}^{-m}, t_{2, i+1}^{-m}, \dots) = \frac{1}{p^r} \sum_{m=0}^{p^r-1} \mathbf{t}_i^m \mathbf{t}_{i+1}^{-m}$$

Subtracting, now, from $e_{p^\infty, i}$ each element of the sequence $(\mathbf{e}_{p^r, i})_r$ successively, we obtain:

$$\begin{aligned} (e_{p, i}, e_{p^2, i}, e_{p^3, i}, \dots) - (e_{p, i}, \zeta_{2,1, i}, \zeta_{3,1, i}, \dots) &= (0, e_{p^2, i} - \zeta_{2,1, i}, e_{p^3, i} - \zeta_{3,1, i}, \dots) \\ (e_{p, i}, e_{p^2, i}, e_{p^3, i}, \dots) - (\zeta_{1,2, i}, e_{p^2, i}, \zeta_{3,2, i}, \dots) &= (0, 0, e_{p^3, i} - \zeta_{3,2, i}, \dots) \\ &\vdots \end{aligned}$$

showing the approximation of $(\mathbf{e}_{p^r, i})_r$ to $e_{p^\infty, i}$. \square

2.6. Approximating purely p -adic elements. The approximation of $e_{p^\infty, i}$ indicates the method for approximating purely p -adic elements in $\varprojlim_r \mathbb{C}C_r^n$, in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ and in $Y_{p^\infty, n}(u)$. Indeed, we give first the following definition:

Definition 7. Let $\underline{y} = (y_1, y_2, \dots)$ an element in $\varprojlim_r \mathbb{C}\mathcal{F}_{p^r, n}$ resp. in $Y_{p^\infty, n}(u)$, in reduced form according to Definition 3. For any indices r, s with $r \geq s$ we define the element $y_{r, s}$ in $\mathbb{C}\mathcal{F}_{p^r, n}$ resp. $Y_{p^r, n}(u)$ as the formal expression of y_s , but with the generators $t_{s, i}$ replaced by the generators $t_{r, i}$, for all i . The element $y_{r, s}$ is called the s -truncation of y_r .

Further, we define the element $y_{s, r}$ in $\mathbb{C}\mathcal{F}_{p^s, n}$ resp. $Y_{p^s, n}(u)$ as the formal expression of y_r , but with the generators $t_{r, i}$ replaced by the generators $t_{s, i}$, for all i . The element $y_{s, r}$ is called the r -expansion of y_s .

Note that, either way, $y_{r, r} = y_r$. We define now the element \mathbf{y}_r by r -truncating each term in \underline{y} after the r th position and by r -expanding each term before the r th position. That is,

$$\mathbf{y}_r = (y_{1, r}, y_{2, r}, y_r, y_{r+1, r}, y_{r+2, r}, \dots)$$

Theorem 2. For any index r the element \mathbf{y}_r is a constant sequence in $\varprojlim_r \mathbb{C}\mathcal{F}_n = \mathbb{C}\mathcal{F}_n$ resp. in $\mu(\mathbb{C}\mathcal{F}_n)$. Moreover we have the approximation:

$$(2.8) \quad \underline{y} = \lim_r \mathbf{y}_r$$

In this sense, \underline{x} can be viewed as an infinite series:

$$\underline{y} = \sum_{r=0}^{\infty} \mathbf{y}_r$$

Proof. Since now $\varphi_s^r(y_r) = y_s$ it must be also true, by construction, that $\varphi_s^r(y_{r, s}) = y_s$, that is, the elements $y_{r, s}$ and y_s are coherent. In fact, $y_{r, s}$ 'wraps' only once on y_s via the map φ_s^r . Moreover, since $\varphi_s^r(y_r) = y_s$ it must be also true, by construction, that $\varphi_s^r(y_r) = y_{s, r}$, so $y_{s, r} = y_s$. Completely analogous thoughts for ϕ_s^r in place of φ_s^r . The element \mathbf{y}_r is by construction a constant sequence. To see now that these constant sequences approximate \underline{y} we subtract them successively from \underline{y} and we confirm that the zero-sequence is gradually forming. \square

3. A TOPOLOGICAL MARKOV TRACE

In [3] the first author constructed linear Markov traces on the Yokonuma–Hecke algebras. The aim of this section is to extend these traces to a p -adic Markov trace on the algebra $Y_{p^\infty, n}(u)$.

3.1. A Markov trace on the Yokonuma–Hecke algebra. The natural inclusions $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ of the classical framed braid groups induce natural inclusions $\mathcal{F}_{d,n} \subset \mathcal{F}_{d,n+1}$ of modular framed braid groups. These, in turn, induce the algebra inclusions:

$$\mathbb{C}\mathcal{F}_{d,0} \subset \mathbb{C}\mathcal{F}_{d,1} \subset \mathbb{C}\mathcal{F}_{d,2} \subset \dots$$

(setting $\mathbb{C}\mathcal{F}_{d,0} := \mathbb{C}$), which in turn induce the tower of algebras:

$$(3.1) \quad Y_{d,0}(u) \subset Y_{d,1}(u) \subset Y_{d,2}(u) \subset \dots$$

(setting $Y_{d,0}(u) := \mathbb{C}$). Thus, given d , we have the inductive system $(Y_{d,n}(u))_{n \in \mathbb{N}}$. Let $Y_{d,\infty}(u)$ be the corresponding inductive limit. Then we have the following.

Theorem 3 (cf. Theorem 12 in [3]). *Let d a positive integer. For indeterminates z, x_1, \dots, x_{d-1} there exists a unique linear Markov trace $\text{tr}_d = (\text{tr}_{d,n})_{n \in \mathbb{N}}$*

$$\text{tr}_d : Y_{d,\infty}(u) \longrightarrow \mathbb{C}[z, x_1, \dots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{aligned} \text{tr}_{d,n}(ab) &= \text{tr}_{d,n}(ba) \\ \text{tr}_{d,n}(1) &= 1 \\ \text{tr}_{d,n}(ag_n) &= z \text{tr}_{d,n}(a) && (\text{Markov property}) \\ \text{tr}_{d,n+1}(at_{n+1}^m) &= x_m \text{tr}_{d,n}(a) && (m = 1, \dots, d-1) \end{aligned}$$

where $a, b \in Y_{d,n}(u)$.

Diagrammatically, in the second rule is meant the trace of the identity braid of any number of strands with all framings zero. The third rule is the so-called *Markov property* of the trace. View Figure 6 for topological interpretations of the last two rules.

$$\text{tr}_d \left(\begin{array}{c} \text{box } a \\ \text{crossing} \end{array} \right) = z \text{tr}_d \left(\begin{array}{c} \text{box } a \end{array} \right), \quad \text{tr}_d \left(\begin{array}{c} \text{box } a \\ \text{vertical line } m \end{array} \right) = x_m \text{tr}_d \left(\begin{array}{c} \text{box } a \end{array} \right)$$

FIGURE 6. Topological interpretations of the trace rules

The key in the construction of tr_d is that $Y_{d,n+1}(u)$ has a ‘nice’ inductive linear basis. Indeed, every element of $Y_{d,n+1}(u)$ is a unique linear combination of words, each of one of the following types:

$$(3.2) \quad w_n g_n g_{n-1} \dots g_i t_i^k \quad \text{or} \quad w_n t_{n+1}^k, \quad k \in \mathbb{Z}/d\mathbb{Z}$$

where $w_n \in Y_{d,n}(u)$. Thus, the above words furnish an inductive basis for $Y_{d,n+1}(u)$, every element of which involves g_n or a power of t_{n+1} at most once. For a proof see [3].

Remark 3. In the case $d = 1$, when the algebra $Y_{1,n}(u)$ coincides with the Iwahori–Hecke algebra $H_n(u)$, the trace tr_1 coincides with the Ocneanu trace, cf. [2].

3.2. The p -adic Markov trace. Let, now, R denote the polynomial ring $\mathbb{C}[z]$ and let r be a positive integer. We denote $R[\mathfrak{X}_r]$ the polynomial ring on the indeterminates of the set \mathfrak{X}_r ,

$$\mathfrak{X}_r := \{x_a; a \in \mathbb{Z}/p^r\mathbb{Z}\}$$

For all positive integers r, s such that $r \geq s$ we have the ring homomorphism

$$(3.3) \quad \delta_s^r : R[\mathfrak{X}_r] \longrightarrow R[\mathfrak{X}_s]$$

which is defined through the map $x_a \mapsto x_b$, where $b := \vartheta_s^r(a)$. It is a routine to prove the following lemma.

Lemma 6. *The family $(R[\mathfrak{X}_r], \delta_s^r)$ is an inverse system of polynomial rings indexed by \mathbb{N} .*

Notations. We shall denote τ_r in place of tr_{p^r} and $\tau_{r,n}$ in place of $\text{tr}_{p^r,n}$. With these notations: $\tau_r = (\tau_{r,n})_{n \in \mathbb{N}}$.

Lemma 7. *The diagram below is commutative.*

$$\begin{array}{ccc} Y_{p^r,n}(u) & \xrightarrow{\phi_s^r} & Y_{p^s,n}(u) \\ \tau_{r,n} \downarrow & & \downarrow \tau_{s,n} \\ R[\mathfrak{X}_r] & \xrightarrow{\delta_s^r} & R[\mathfrak{X}_s] \end{array}$$

Proof. The proof is by induction on n . The lemma is immediate for $n = 1$. Assume the lemma is true for some n . In order to prove it for $n + 1$ we must check that $(\tau_{s,n+1} \circ \phi_s^r)(x) = (\delta_s^r \circ \tau_{r,n+1})(x)$ for all $x \in Y_{p^r,n+1}(u)$. Since, by definition, $\phi_s^r, \tau_{s,n+1}$ and $\tau_{r,n+1}$ are linear maps, it suffices to prove that $(\tau_{s,n+1} \circ \phi_s^r)(\alpha) = (\delta_s^r \circ \tau_{r,n+1})(\alpha)$, for α in the inductive basis of $Y_{p^r,n+1}(u)$. Firstly, assume

$\alpha = w_n g_n g_{n-1} \dots g_i t_{r,i}^k$, where $w_n \in Y_{p^r, n-1}(u)$. Then:

$$\begin{aligned}
 \tau_{s,n}(\phi_s^r(\alpha)) &= \tau_{s,n}(\phi_s^r(w_n) g_n g_{n-1} \dots g_i t_{r,i}^k) \quad (k \text{ regarded modulo } p^s) \\
 &= z \tau_{s,n}(\phi_s^r(w_n) g_{n-1} \dots g_i t_{r,i}^k) \\
 &= z \tau_{s,n}(\phi_s^r(w_n g_{n-1} \dots g_i t_{r,i}^k)) \\
 &= z \delta_s^r(\tau_{r,n}(w_n g_{n-1} \dots g_i t_{r,i}^k)) \quad (\text{induction hypothesis}) \\
 &= \delta_s^r(z \tau_{r,n}(w_n g_{n-1} \dots g_i t_{r,i}^k)) \\
 &= \delta_s^r(\tau_{r,n}(w_n g_n g_{n-1} \dots g_i t_{r,i}^k)) \quad (\text{trace rule}) \\
 &= \delta_s^r(\tau_{r,n}(\alpha)).
 \end{aligned}$$

Secondly, assume $\alpha = w_n t_{r,n}^k$. Then:

$$\tau_{s,n}(\phi_s^r(\alpha)) = \tau_{s,n}(\phi_s^r(w_n) t_{r,n}^k) = x_k \tau_{s,n}(\phi_s^r(w_n))$$

(note that k is regarded modulo p^s). By the induction hypothesis we obtain:

$$\tau_{s,n}(\phi_s^r(\alpha)) = x_k \delta_s^r(\tau_{r,n}(w_n)) = \delta_s^r(x_k \tau_{r,n}(w_n))$$

and by the trace rule:

$$\delta_s^r(x_k \tau_{r,n}(w_n)) = \delta_s^r(\tau_{r,n}(w_n t_{r,n+1}^k)) = \delta_s^r(\tau_{r,n}(\alpha))$$

where now k is regarded modulo p^r . Hence the proof follows. \square

Definition 8. For a p -adic integer $\underline{a} = (a_1, a_2, \dots) \neq 0$ we shall denote

$$x_{\underline{a}} := (x_{a_1}, x_{a_2}, \dots) \in \varprojlim_{r \in \mathbb{N}} R[\mathfrak{X}_r]$$

and we shall call $x_{\underline{a}}$ a *p -adic indeterminate*. Further, for an almost constant sequence $a_i = (a_1, \dots, a_{i-1}, a_i, a_i, \dots)$ in \mathbb{Z} we shall denote

$$x_{a_i} := (x_{a_1}, \dots, x_{a_{i-1}}, x_{a_i}, x_{a_i}, \dots) \in \varprojlim_{r \in \mathbb{N}} R[\mathfrak{X}_r]$$

and we shall say that x_{a_i} is a *constant indeterminate*. Finally, we make the convention $x_0 := 1$.

Let r, s and v be positive integers such that $r \geq s \geq v$. By Lemmas 6 and 7, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \cdots & \longleftarrow & Y_{p^v, n}(u) & \xleftarrow{\rho_v^s} & Y_{p^s, n}(u) & \xleftarrow{\rho_s^r} & Y_{p^r, n}(u) & \longleftarrow \cdots \\
 & & \downarrow \tau_{v, n} & & \downarrow \tau_{s, n} & & \downarrow \tau_{r, n} & \\
 \cdots & \longleftarrow & R[\mathfrak{X}_v] & \xleftarrow{\delta_v^s} & R[\mathfrak{X}_s] & \xleftarrow{\delta_s^r} & R[\mathfrak{X}_r] & \longleftarrow \cdots
 \end{array}$$

The diagram above induces a unique ring homomorphism $\tau_{p^\infty, n} := \varprojlim_r \tau_{r, n}$,

$$\tau_{p^\infty, n} : Y_{p^\infty, n}(u) \longrightarrow \varprojlim_{r \in \mathbb{N}} R[\mathfrak{X}_r]$$

Recall that, by definition, $\tau_{p^\infty, n} = (\tau_{1, n}, \tau_{2, n}, \tau_{3, n}, \dots)$.

On the other hand, the inclusions (3.1) of Yokonuma–Hecke algebras induce, by construction, an inductive system of p -adic Yokonuma–Hecke algebras:

$$Y_{p^\infty, 0}(u) \subset Y_{p^\infty, 1}(u) \subset Y_{p^\infty, 2}(u) \subset \dots$$

(setting $Y_{p^\infty, 0}(u) := \mathbb{C}$). Let $Y_{p^\infty, \infty}(u)$ be the associated inductive limit:

$$Y_{p^\infty, \infty}(u) := \varinjlim_{n \in \mathbb{N}} Y_{p^\infty, n}(u)$$

Theorem 4. *There exists a unique linear Markov trace $\tau_{p^\infty} = (\tau_{p^\infty, n})_{n \in \mathbb{N}}$,*

$$\tau_{p^\infty} : Y_{p^\infty, \infty}(u) \longrightarrow \varprojlim_{r \in \mathbb{N}} R[\mathfrak{X}_r]$$

such that:

$$\begin{aligned} \tau_{p^\infty, n}(\underline{\underline{a}} \underline{\underline{b}}) &= \tau_{p^\infty, n}(\underline{\underline{b}} \underline{\underline{a}}) \\ \tau_{p^\infty, n}(1) &= 1 \\ \tau_{p^\infty, n+1}(\underline{\underline{y}} \underline{\underline{g}}_n) &= z \tau_{p^\infty, n}(\underline{\underline{y}}) && (\text{Markov property}) \\ \tau_{p^\infty, n+1}(\underline{\underline{y}} \underline{\underline{\mathbf{t}}}_{n+1}^m) &= x_m \tau_{p^\infty, n}(\underline{\underline{y}}) && (m \in \mathbb{Z}) \\ \tau_{p^\infty, n+1}(\underline{\underline{y}} \underline{\underline{\mathbf{t}}}_{n+1}^{\underline{\underline{m}}}) &= x_{\underline{\underline{m}}} \tau_{p^\infty, n}(\underline{\underline{y}}) && (\underline{\underline{m}} \in \mathbb{Z}_p) \end{aligned}$$

where $\underline{\underline{a}}, \underline{\underline{b}}, \underline{\underline{y}} \in Y_{p^\infty, n}$.

Proof. The trace τ_{p^∞} is unique by Theorem 3 and, by construction, it satisfies all properties in the statement. Indeed, let us check the third and the fifth property. For $\underline{\underline{y}} = (y_r)_r$ with $y_r \in Y_{p^r, n}(u)$ and $\underline{\underline{m}} = (m_r)_r \in \mathbb{Z}_p$ we have $\tau_{p^\infty, n}(\underline{\underline{y}} \underline{\underline{g}}_n) = (\tau_{r, n}(y_r g_n))_r$. Hence, using Theorem 3,

$$\tau_{p^\infty, n}(\underline{\underline{y}} \underline{\underline{g}}_n) = (z \tau_{r, n}(y_r))_r z (\tau_{r, n}(y_r))_r = z \tau_{p^\infty, n}(\underline{\underline{y}}).$$

Analogously, we have $\tau_{p^\infty, n}(\underline{\underline{y}} \underline{\underline{\mathbf{t}}}_{n+1}^m) = (\tau_{r, n}(y_r t_{r, n+1}^{m_r}))_r$, then:

$$\tau_{p^\infty, n}(\underline{\underline{y}} \underline{\underline{\mathbf{t}}}_{n+1}^{\underline{\underline{m}}}) = (x_{m_r} \tau_{r, n}(y_r))_r = (x_{m_r})_r (\tau_{r, n}(y_r))_r = x_{\underline{\underline{m}}} \tau_{p^\infty, n}(\underline{\underline{y}}).$$

□

Notation. We shall call τ_{p^∞} a p -adic Markov trace.

Remark 4. Notice that we have the approximation:

$$\tau_{p^\infty, n}(\underline{\underline{y}}) = \lim_r \tau_{p^\infty, n}(\mathbf{y}_r)$$

for $\underline{\underline{y}} = (y_r)_r = \lim_r \mathbf{y}_r \in Y_{p^\infty, n}$. In this sense $\tau_{p^\infty, n}(\underline{\underline{y}})$ can be viewed as an infinite series in $\varprojlim R[\mathfrak{X}_r]$:

$$\tau_{p^\infty, n}(\underline{\underline{y}}) = \sum_{r=0}^{\infty} \tau_{p^\infty, n}(\mathbf{y}_r).$$

Indeed, the approximating sequence to $\tau_{p^\infty, n}(\underline{y})$ by the values of $\tau_{p^\infty, n}$ on the elements $\mathbf{y}_r \in \mu(\mathbb{C}\mathcal{F}_n)$ follows easily by Theorem 2 and our usual approximation arguments.

3.3. Computations. We shall now give some computations of the traces tr_d and τ_{p^∞} .

- For the element $\mathbf{t}^k \in Y_{p^\infty, 1}(u)$ (1-strand braid with framing k) we have:

$$\tau_{p^\infty}(\mathbf{t}^k) = (\tau_1(t_1^{k(\bmod p)}), \tau_2(t_2^{k(\bmod p^2)}), \dots) = (x_k, x_k, \dots) = x_k \in R[\mathfrak{X}_r]$$

which follows from Definition 8.

- For the element $\mathbf{t}^{\underline{a}} = (t_1^{a_1}, t_2^{a_2}, \dots) \in Y_{p^\infty, 1}(u)$, where $\underline{a} = (a_r)_r$, we have:

$$\tau_{p^\infty}(\mathbf{t}^{\underline{a}}) = (\tau_1(t_1^{a_1}), \tau_2(t_2^{a_2}), \dots) = (x_{a_1}, x_{a_2}, \dots) = x_{\underline{a}} \in \varprojlim R[\mathfrak{X}_r].$$

Further, since $\mathbf{t}^{\underline{a}} = \lim_r \mathbf{t}^{a_r}$ we have the following trace approximation:

$$\tau_{p^\infty}(\mathbf{t}^{\underline{a}}) = \lim_r \tau_{p^\infty}(\mathbf{t}^{a_r}) = \lim_r x_{a_r} \in \varprojlim R[\mathfrak{X}_r]$$

- For the n -strand identity braid with framings $k_1, \dots, k_n \in \mathbb{Z}$ we have:

$$\text{tr}_d(t_1^{k_1} \dots t_n^{k_n}) = x_{k_1} \dots x_{k_n}$$

$$\tau_{p^\infty}(\mathbf{t}_1^{k_1} \dots \mathbf{t}_n^{k_n}) = (\tau_{r,n}(t_{r,1}^{k_1} \dots t_{r,n}^{k_n}))_r = x_{k_1} \dots x_{k_n} = \tau_{p^\infty}(\mathbf{t}_1^{k_1}) \dots \tau(\mathbf{t}_n^{k_n}) \in R[\mathfrak{X}_r]$$

- For the n -strand identity braid with framings $\underline{a}_i = (a_{ri})_r \in \mathbb{Z}_p$ we have:

$$\begin{aligned} \tau_{p^\infty}(\mathbf{t}_1^{\underline{a}_1} \dots \mathbf{t}_n^{\underline{a}_n}) &= (\tau_{r,n}(t_{r,1}^{a_{r1}} \dots t_{r,n}^{a_{rn}}))_r \\ &= (x_{a_{11}} \dots x_{a_{1n}}, x_{a_{21}} \dots x_{a_{2n}}, \dots) \\ &= (x_{a_{11}}, x_{a_{21}}, \dots) \dots (x_{a_{1n}}, x_{a_{2n}}, \dots) \\ &= x_{\underline{a}_1} \dots x_{\underline{a}_n} \\ &= \tau_{p^\infty}(\mathbf{t}_1^{\underline{a}_1}) \dots \tau(\mathbf{t}_n^{\underline{a}_n}) \in \varprojlim R[\mathfrak{X}_r]. \end{aligned}$$

Further, we have the approximation:

$$\tau_{p^\infty}(\mathbf{t}_1^{\underline{a}_1} \dots \mathbf{t}_n^{\underline{a}_n}) = \lim_r \tau_{p^\infty}(\mathbf{t}_1^{a_{r1}} \dots \mathbf{t}_n^{a_{rn}}) = \lim_r (x_{a_{r1}} \dots x_{a_{rn}}) \in \varprojlim R[\mathfrak{X}_r].$$

- For the elements $e_{d,i} \in Y_{d,n}(u)$ and $e_{p^\infty, i} \in Y_{p^\infty, n}(u)$ ($i = 1, \dots, n-1$) :

$$(3.4) \quad E_d := \text{tr}_d(e_{d,i}) = \text{tr}_d \left(\frac{1}{d} \sum_{m=0}^{d-1} t_i^m t_{i+1}^{-m} \right) = \frac{1}{d} \sum_{m=0}^{d-1} x_m x_{d-m}$$

$$\tau_{p^\infty}(e_{p^\infty, i}) = \tau_{p^\infty}((e_{p,i}, e_{p^2, i}, \dots)) = (\tau_{r,n}(e_{p^r, i}))_r = (E_p, E_{p^2}, \dots) = (E_{p^r})_r \in \varprojlim R[\mathfrak{X}_r]$$

Further, from (2.8) we have the approximation:

$$\tau_{p^\infty}(e_{p^\infty,i}) = \lim_r \tau_{p^\infty}(\mathbf{e}_{p^r,i}) = \lim_r \left(\frac{1}{p^r} \sum_{m=0}^{p^r-1} x_m x_{-m} \right) = \lim_r E_{p^r} \in \varprojlim R[\mathfrak{X}_r]$$

- For the elements $e_{d,i}g_i \in Y_{d,n}(u)$ and $e_{p^\infty,i}g_i \in Y_{p^\infty,n}(u)$ ($i = 1, \dots, n-1$) we have the following more general lemma.

Lemma 8. *Let $y \in Y_{d,n}(u)$ and set $\underline{y} = (y_r)_r = \lim_r \mathbf{y}_r \in Y_{p^\infty,n}(u)$. Then we have:*

- (1) $\mathrm{tr}_d(y e_{d,n}g_n) = \mathrm{tr}_d(yg_n) = z \mathrm{tr}_d(y)$
- (2) $\tau_{p^\infty}(\underline{y} e_n g_n) = \tau_{p^\infty}(\underline{y} g_n) = z \tau_{p^\infty}(\underline{y})$
- (3) $\tau_{p^\infty}(\underline{y} e_n g_n) = z \lim_r \tau_{p^\infty}(\mathbf{y}_r)$.

Proof. (i) We have $y e_{d,n}g_n = \frac{1}{d} \sum_{m=0}^{d-1} y t_n^m t_{n+1}^{-m} g_n = \frac{1}{d} \sum_{m=0}^{d-1} y t_n^m g_n t_n^{-m}$ so, applying the trace rule yields the statement.

(ii) $\tau_{p^\infty}(\underline{y} e_n g_n) = (\tau_{r,n}(y_r e_{p^r,n}g_n))_r \stackrel{(i)}{=} (z \tau_{r,n}(y_r))_r = z \tau_{p^\infty}(\underline{y}) = \tau_{p^\infty}(\underline{y} g_n)$.

Finally, (iii) follows immediately from (ii) and Theorem 2. \square

- For $g_i^2 \in Y_{d,n}(u)$ and for $g_i^2 \in Y_{\infty,n}$, where $g_i^2 = (g_i^2, g_i^2, \dots)$ we have:

$$\mathrm{tr}_d(g_i^2) = 1 - (u-1)z + (u-1)E_d$$

$$\tau_{p^\infty}(g_i^2) = 1 - (u-1)z + (u-1)(E_{p^r})_r = 1 - (u-1)z + (u-1)\tau_{p^\infty}(e_{p^\infty,i}) \in \varprojlim R[\mathfrak{X}_r]$$

- Finally, we have:

$$\mathrm{tr}_d(g_i^3) = (u^2 - u + 1)z + (u^2 - u)E_d.$$

$$\mathrm{tr}_d(g_i^{-3}) = (u^{-3} - u^{-2} + u^{-1})z - (u^{-3} - u^{-2} + u^{-1} - 1)E_d.$$

$$\tau_{p^\infty}(g_i^3) = (u^2 - u + 1)z + (u^2 - u)(E_{p^r})_r \in \varprojlim R[\mathfrak{X}_r].$$

$$\tau_{p^\infty}(g_i^{-3}) = (u^{-3} - u^{-2} + u^{-1})z - (u^{-3} - u^{-2} + u^{-1} - 1)(E_{p^r})_r \in \varprojlim R[\mathfrak{X}_r].$$

3.4. The Markov braid equivalence. From the topological point of view, closing a framed braid gives rise to an oriented *framed link* and closing a p -adic framed braid gives rise to an oriented p -adic *framed link*, view Figure 2. By ‘closing’ a braid β we mean the standard closure, denoted $\widehat{\beta}$, where we join with simple arcs the corresponding top and bottom endpoints of the braid. Conversely, by the classical Alexander theorem (adapted to the various framed braid settings), an oriented framed link can be isotoped to the closure of a framed braid.

Further, by the classical Markov theorem (adapted to the various framed braid settings), isotopy classes of oriented framed links are in one-to-one correspondence with equivalence classes of framed braids. More precisely, the

natural inclusions $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\mathcal{F}_{d,n} \subset \mathcal{F}_{d,n+1}$ of the classical and the modular framed braid groups and the induced natural inclusions $\mathcal{F}_{p^\infty,n} \subset \mathcal{F}_{p^\infty,n+1}$ of the p -adic framed braid groups give rise to the direct limits \mathcal{F}_∞ , $\mathcal{F}_{d,\infty}$ and $\mathcal{F}_{p^\infty,\infty}$ respectively. Then we have the following result, which is well-known for the case of classical framed links (see for example [7]), and which we also adapt here for the cases of modular framed links and p -adic framed links.

Theorem 5 (Markov equivalence for framed braids and p -adic framed braids). *Isotopy classes of oriented framed links (resp. modular framed links) are in bijection with equivalence classes of framed braids in \mathcal{F}_∞ (resp. $\mathcal{F}_{d,\infty}$). The equivalence relation is generated by the two moves:*

- (i) *Conjugation:* $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in \mathcal{F}_n$ (resp. $\mathcal{F}_{d,n}$)
- (ii) *Markov move:* $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in \mathcal{F}_n$ (resp. $\mathcal{F}_{d,n}$)

Further, isotopy classes of p -adic framed links are in bijection with equivalence classes of p -adic framed braids in $\mathcal{F}_{p^\infty,\infty}$ under the following equivalence relation: Two p -adic framed braids $\underline{\alpha} = (\alpha_r)_r$ and $\underline{\beta} = (\beta_r)_r$ in $\mathcal{F}_{p^\infty,\infty}$ are equivalent if, for each r , α_r and β_r are Markov equivalent in $\mathcal{F}_{d,\infty}$. In view of the isomorphisms (1.10) the Markov equivalence of p -adic framed braids is generated by the moves:

- (i) *Conjugation:* $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in \mathcal{F}_{p^\infty,n}$
- (ii) *Markov move:* $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in \mathcal{F}_{p^\infty,n}$.

According to Theorem 5, any invariant of oriented framed links has to agree on the closures of the braids α , $\alpha\sigma_n$ and $\alpha\sigma_n^{-1}$. Note the resemblance of the conjugation rule and the Markov property in Theorems 3 and 4 with moves (i) and (ii) of Theorem 5. Having, now, present the recipe of Jones[2] we will try to define an invariant by re-scaling and normalization of the trace tr_d and the p -adic trace τ_{p^∞} . In order to do that we need that the expression $\text{tr}_d(\alpha g_n^{-1})$, for $\alpha \in Y_{d,n}(u)$, factors through $\text{tr}_d(\alpha)$, just like $\text{tr}_d(\alpha g_n)$ does from the Markov property of the trace. But

$$\text{tr}_d(\alpha g_n^{-1}) = \text{tr}_d(\alpha g_n) - (u^{-1} - 1)\text{tr}_d(\alpha e_{d,n}) + (u^{-1} - 1)\text{tr}_d(\alpha e_{d,n} g_n)$$

Analogous requirements apply to $\tau_{p^\infty}(\alpha g_n^{-1})$, for $\alpha \in Y_{p^\infty,n}(u)$. Here we have:

$$\tau_{p^\infty}(\alpha g_n^{-1}) = \tau_{p^\infty}(\alpha g_n) - (u^{-1} - 1)\tau_{p^\infty}(\alpha e_n) + (u^{-1} - 1)\tau_{p^\infty}(\alpha e_n g_n)$$

By Lemma 8 (i) and (ii), we only need further that the traces tr_d and τ_{p^∞} satisfy the multiplicative properties:

$$(3.5) \quad \text{tr}_d(\alpha e_{d,n}) = \text{tr}_d(\alpha) \text{tr}_d(e_{d,n}) \quad \alpha \in Y_{d,n}(u)$$

and

$$(3.6) \quad \tau_{p^\infty}(\alpha e_n) = \tau_{p^\infty}(\alpha) \tau_{p^\infty}(e_n) \quad \alpha \in Y_{p^\infty,n}(u)$$

With these properties we could then define framed link invariants using the same method as for defining the Jones polynomial[2]. Unfortunately, we do not have a nice formula for $\text{tr}_d(\alpha e_{d,n})$, and this causes the same problem for $\tau_{p^\infty}(\alpha e_n)$. The reason is that the element $e_{d,n}$ involves the n th strand of the braid α .

4. THE E -CONDITION

The goal of this section is to find conditions, so that equations (3.5) and (3.6) hold. Since d remains fixed throughout the section we shall denote tr_d simply by tr and we will suppress the index d from the framing generators of the algebras $Y_{d,n}(u)$. We shall also suppress the values of the indices in the summation symbols.

For $0 \leq k \leq d-1$ we now define the elements:

$$e_{d,i}^{(k)} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^{k+s} t_{i+1}^{d-s}$$

and also:

$$E_d^{(k)} := \text{tr} \left(e_{d,i}^{(k)} \right) = \frac{1}{d} \sum_{s=0}^{d-1} x_{k+s} x_{d-s}$$

With the above notation $e_{d,i}^{(0)} = e_{d,i}$ and $E_d^{(0)} = E_d = \text{tr}(e_{d,i})$. Note that in the definition of $E_d^{(k)}$ the sub-indices of the indeterminates are regarded modulo d . For example: $E_3^{(2)} = 1/3 (2x_2 + x_1^2)$.

Remark 5. By a change of variable for s it is easy to deduce the following useful formulas, for $k, l \in \mathbb{Z}$, stressing once more that the sub-indices of the indeterminates are regarded modulo d .

$$\frac{1}{d} \sum_{s=0}^{d-1} t_i^{k+s} t_{i+1}^{l-s} = e_{d,i}^{(k+l)} \quad \text{and} \quad \frac{1}{d} \sum_{s=0}^{d-1} x_{k+s} x_{l-s} = E_d^{(k+l)}$$

4.1. *Computing $\text{tr}_d(\alpha e_{d,n})$.* By (3.2) every element in $Y_{d,n}(u)$ is a unique linear combination of words in one of the following types:

$$w_{n-1} g_{n-1} \dots g_i t_i^k \quad \text{or} \quad w_{n-1} t_n^k, \quad k \in \mathbb{Z}/d\mathbb{Z}$$

We shall now give some concrete computations.

- For $n = 1$ the only case is $\alpha = t_1^k$. So: $\text{tr}(\alpha) = x_k$ and $\text{tr}(\alpha e_{d,1}) = E_d^{(k)}$.
- For $n = 2$ we have the following cases for α : (i) $\alpha = t_1^{k_1} t_2^k$ and (ii) $\alpha = t_1^{k_1} g_1 t_1^k$.
 - (i) $\text{tr}(\alpha) = x_{k_1} x_k$ and $\text{tr}(\alpha e_{d,2}) = x_{k_1} E_d^{(k)} = \frac{E_d^{(k)}}{x_k} \text{tr}(\alpha)$.
 - (ii) $\text{tr}(\alpha) = z x_{k_1+k}$ and $\text{tr}(\alpha e_{d,2}) = z E_d^{(k_1+k)} = \frac{E_d^{(k_1+k)}}{x_{k_1+k}} \text{tr}(\alpha)$.

In general we have the following results.

Lemma 9. *Let $\alpha = w_{n-1}t_n^k$ with $w_{n-1} \in Y_{d,n-1}(u)$. Then:*

$$\mathrm{tr}(\alpha e_{d,n}) = \frac{E_d^{(k)}}{x_k} \mathrm{tr}(\alpha).$$

More generally: $\mathrm{tr}(\alpha e_{d,n}^{(m)}) = \frac{E_d^{(m+k)}}{x_k} \mathrm{tr}(\alpha).$

Proof. We prove the more general result. We have:

$$\begin{aligned} \mathrm{tr}(\alpha e_{d,n}^{(m)}) &= \frac{1}{d} \sum_s \mathrm{tr}(w_{n-1} t_n^k t_n^{m+s} t_{n+1}^{-s}) \\ &= \frac{1}{d} \sum_s x_{-s} \mathrm{tr}(w_{n-1} t_n^{m+k+s}) \\ &= \frac{1}{d} \sum_s x_{-s} x_{m+k+s} \mathrm{tr}(w_{n-1}) \\ &= \mathrm{tr}(w_{n-1}) \frac{1}{d} \sum_s x_{-s} x_{m+k+s} = \mathrm{tr}(w_{n-1}) E_d^{(m+k)} \end{aligned}$$

On the other hand $\mathrm{tr}(\alpha) = x_k \mathrm{tr}(w_{n-1})$. □

Lemma 10. *Let $\alpha = w_{n-1}g_{n-1} \cdots g_i t_i^k \in Y_{d,n}(u)$, with $1 \leq i < n$ and $w_{n-1} \in Y_{d,n-1}(u)$. Then we have:*

$$\mathrm{tr}(\alpha e_{d,n}) = z \mathrm{tr}(\alpha' e_{d,n-1})$$

where $\alpha' := g_{n-2} \cdots g_i t_i^k w_{n-1} \in Y_{d,n-1}(u)$.

Proof. We have:

$$\begin{aligned} \mathrm{tr}(\alpha e_{d,n}) &= \frac{1}{d} \sum_s x_{-s} \mathrm{tr}(w_{n-1} g_{n-1} \cdots g_i t_i^k t_n^s) \\ &= \frac{1}{d} \sum_s x_{-s} \mathrm{tr}(w_{n-1} t_{n-1}^s g_{n-1} \cdots g_i t_i^k) \\ &= \frac{z}{d} \sum_s x_{-s} \mathrm{tr}(w_{n-1} t_{n-1}^s g_{n-2} \cdots g_i t_i^k) \\ &= \frac{z}{d} \sum_s x_{-s} \mathrm{tr}(g_{n-2} \cdots g_i t_i^k w_{n-1} t_{n-1}^s) \\ &= \frac{z}{d} \sum_s x_{-s} \mathrm{tr}(\alpha' t_{n-1}^s) = \frac{z}{d} \sum_s \mathrm{tr}(\alpha' t_{n-1}^s t_n^{-s}) = z \mathrm{tr}(\alpha' e_{d,n-1}). \end{aligned}$$

□

• For $n = 3$ we have the following possibilities for α :

$$(i) \ t_1^{k_1} t_2^{k_2} t_3^k \quad (ii) \ t_1^{k_1} t_2^{k_2} g_2 t_2^k \quad (iii) \ t_1^{k_1} t_2^{k_2} g_2 g_1 t_1^k$$

$$(iv) \quad g_1 t_1^{k_1} t_3^k \quad (v) \quad g_1 t_1^{k_1} g_2 t_2^k \quad (vi) \quad g_1 t_1^{k_1} g_2 g_1 t_1^k$$

Cases (i) and (iv) are applications of Lemma 9. Cases (ii), (iii) and (v) show also factorizing through $\text{tr}(\alpha)$. Indeed, by direct computations we obtain:

$$(ii) \quad \text{tr}(\alpha e_{d,3}) = \frac{E_d^{(k_2+k)}}{x_{k_2+k}} \text{tr}(\alpha)$$

$$(iii) \quad \text{tr}(\alpha e_{d,3}) = \frac{E_d^{(k_1+k_2+k)}}{x_{k_1+k_2+k}} \text{tr}(\alpha)$$

$$(v) \quad \text{tr}(\alpha e_{d,3}) = \frac{E_d^{(k_1+k)}}{x_{k_1+k}} \text{tr}(\alpha)$$

Notice that, even in the above simple cases, where $\text{tr}(\alpha e_{d,3})$ factors through $\text{tr}(\alpha)$, the factors are different in all examples. It remains now to consider case (vi) for α . Indeed we have:

$$\begin{aligned} (vi) \quad \text{tr}(\alpha) &= z \text{tr}(g_1^2 t_2^{k_1} t_1^k) \\ &\stackrel{(1.4)}{=} z x_{k_1} x_k + z(u-1) E_d^{(k_1+k)} - z^2(u-1) x_{k_1+k}. \\ \text{tr}(\alpha e_{d,3}) &= \frac{z}{d} \sum_s x_{-s} \text{tr}(g_1^2 t_2^{k_1} t_1^{k+s}) \\ &\stackrel{(1.4)}{=} z x_{k_1} E_d^{(k)} + \frac{z(u-1)}{d} \sum_s x_{-s} E_d^{(k+k_1+s)} - z^2(u-1) E_d^{(k_1+k)}. \end{aligned}$$

It is clear from the above that in order to have Eq. 3.5 we must impose conditions on the set of indeterminates $X_d = \{x_1, x_2, \dots, x_{d-1}\}$.

For example we have the following:

Lemma 11. *Let $c \in \mathbb{C} \setminus \{0\}$. Setting $x_i = c^i$, we have:*

$$\text{tr}(\alpha e_{d,n}) = \text{tr}(\alpha) \text{tr}(e_{d,n}) \quad \text{and} \quad \text{tr}(e_{d,n}) = 1 \quad (\alpha \in Y_{d,n}(u))$$

Proof. The second equality follows immediately by a direct computation. We shall prove the first one by induction. For $n = 1$ we have from the above: $\text{tr}(\alpha) = c^k$ and $\text{tr}(\alpha e_{d,1}) = E_d^{(k)} = \frac{1}{d} \sum_{s=0}^{d-1} c^{k+s} c^{-s} = c^k$. Suppose the statement is true for any element in $Y_{d,n-1}(u)$.

Let now α be an element of the inductive basis of $Y_{d,n}(u)$. If $\alpha = w_{n-1} t_n^k$, where $w_{n-1} \in Y_{d,n-1}(u)$ then, by Lemma 9: $\text{tr}(\alpha e_{d,n}) = \frac{E_d^{(k)}}{x_k} \text{tr}(\alpha) = \frac{c^k}{c^k} \text{tr}(\alpha)$. If, finally, $\alpha = g_{n-1} \dots g_i t_i^k w_{n-1} \in Y_{d,n}(u)$, with $w_{n-1} \in Y_{d,n-1}(u)$ then, by Lemma 10: $\text{tr}(\alpha e_{d,n}) = z \text{tr}(\alpha' e_{d,n-1})$, where $\alpha' = g_{n-2} \dots g_i t_i^k w_{n-1}$ in $Y_{d,n-1}(u)$. Using now the induction hypothesis on the canonical word α' we obtain: $\text{tr}(\alpha e_{d,n}) = z \text{tr}(\alpha') \text{tr}(e_{d,n-1}) = \text{tr}(g_{n-1} \alpha') \text{tr}(e_{d,n}) = \text{tr}(\alpha) \text{tr}(e_{d,n})$. \square

Unfortunately, the condition $x_i = c^i$ does not lead to an interesting invariant from the topological viewpoint. For example:

$$\mathrm{tr}_d(t_1^k t_2^l) = x_k x_l = c^{k+l} = x_{l+k} = \mathrm{tr}_d(t_1^{k+l} t_2^0)$$

but the closures of these two 2-stranded braids are not isotopic as framed (un)links of two components.

4.2. *The E -system.* We shall now seek conditions on a subset of $d - 1$ non-zero complex number other than those of Lemma 11, so that (3.5) and (3.6) are satisfied.

Definition 9. For $m = 0, \dots, d - 1$, let $E_d^{(m)}$ denote the polynomial

$$(4.1) \quad E_d^{(m)} = \sum_{s=0}^{d-1} x_{m+s} x_{d-s}$$

where, by definition $x_0 = x_d = 1$, and the sub-indices are regarded module d . We say that the set of complex numbers $X_d = \{x_1, \dots, x_{d-1}\}$ satisfies the E -condition if x_1, \dots, x_{d-1} satisfy the following E -system of non-linear equations in \mathbb{C} :

$$\begin{aligned} E_d^{(1)} &= x_1 E_d^{(0)} \\ E_d^{(2)} &= x_2 E_d^{(0)} \\ &\vdots \\ E_d^{(d-1)} &= x_{d-1} E_d^{(0)} \end{aligned}$$

Equivalently:

$$(4.2) \quad \sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \quad (1 \leq m \leq d - 1)$$

When two expressions are equal under the E -condition we shall use the notation $\stackrel{E}{=}$. Note that if X_d satisfies the E -system then:

$$\frac{E_d^{(m)}}{x_m} = \frac{E_d^{(k)}}{x_k} = E_d = \mathrm{tr}_d(e_{d,i}) \quad (1 \leq m, k \leq d - 1)$$

Clearly, the E -condition guarantees a common factor, namely $E_d = \mathrm{tr}(e_{d,3})$, at least for the cases where $\mathrm{tr}(\alpha e_{d,n})$ factors through $\mathrm{tr}(\alpha)$. Surprisingly, we also have the following result.

Theorem 6. *If X_d satisfies the E -condition then for all $\alpha \in Y_{d,n}(u)$ we have:*

$$\mathrm{tr}(\alpha e_{d,n}) = \mathrm{tr}(\alpha) \mathrm{tr}(e_{d,n})$$

Proof. By the linearity of the trace it suffices to consider the case when α is an element in the inductive basis of $Y_{d,n}(u)$, recall (3.2). We proceed by induction on n . For $n = 1$ we have: $\text{tr}(\alpha e_{d,1}) = \frac{E_d^{(k)}}{x_k} \text{tr}(\alpha) \stackrel{E}{=} E_d \text{tr}(\alpha) = \text{tr}(\alpha) \text{tr}(e_{d,1})$. Suppose the statement is true for $n - 1$, that is, for all elements in $Y_{d,n-1}(u)$, and let α be an element of the inductive basis of $Y_{d,n}(u)$. If $\alpha = w_{n-1} t_n^k$, where $w_{n-1} \in Y_{d,n-1}(u)$ then, by Lemma 9, we have:

$$\text{tr}(\alpha e_{d,n}) = \frac{E_d^{(k)}}{x_k} \text{tr}(\alpha) \stackrel{E}{=} \text{tr}(\alpha) \text{tr}(e_{d,n})$$

If, finally, $\alpha = g_{n-1} \cdots g_i t_i^k w_{n-1} \in Y_{d,n}(u)$, with $w_{n-1} \in Y_{d,n-1}(u)$ then, by Lemma 10, we have: $\text{tr}(\alpha e_{d,n}) = z \text{tr}(\alpha' e_{d,n-1})$, where $\alpha' = g_{n-2} \cdots g_i t_i^k w_{n-1}$ in $Y_{d,n-1}(u)$. Using now the induction hypothesis on the canonical word α' we obtain: $\text{tr}(\alpha e_{d,n}) \stackrel{E}{=} z \text{tr}(\alpha') \text{tr}(e_{d,n-1}) = \text{tr}(g_{n-1} \alpha') \text{tr}(e_{d,n}) = \text{tr}(\alpha) \text{tr}(e_{d,n})$. \square

Next, we give a useful computational result using the E -condition.

Lemma 12. *For the set of indeterminates X_d we have:*

$$\frac{x_k}{d} \sum_s x_{-s} E_d^{(k+s)} \stackrel{E}{=} \left[E_d^{(k)} \right]^2 \quad (k \in \mathbb{N}).$$

Equivalently,

$$x_k \text{tr} \left(e_{d,n+1} e_{d,n}^{(k)} \right) \stackrel{E}{=} \left[\text{tr} \left(e_{d,n}^{(k)} \right) \right]^2.$$

Proof. Indeed:

$$\frac{1}{d} \sum_s x_{-s} E_d^{(k+s)} \stackrel{E}{=} \frac{1}{d} \sum_s x_{-s} x_{k+s} E_d = E_d E_d^{(k)} \stackrel{E}{=} x_k^{-1} \left[E_d^{(k)} \right]^2.$$

\square

Let us now see how exactly the E -condition works in the case (vi) of $n = 3$, namely when $\alpha = g_1 t_1^{k_1} g_2 g_1 t_1^k$. Recall:

$$\text{tr}(\alpha) = z x_{k_1} x_k + z(u-1) E_d^{(k_1+k)} - z^2(u-1) x_{k_1+k}.$$

Hence:

$$\text{tr}(\alpha) \text{tr}(e_{d,3}) = z x_{k_1} x_k E_d + z(u-1) E_d^{(k_1+k)} E_d - z^2(u-1) x_{k_1+k} E_d.$$

On the other hand:

$$\text{tr}(\alpha e_{d,3}) = z x_{k_1} E_d^{(k)} + \frac{z(u-1)}{d} \sum_s x_{-s} E_d^{(k+k_1+s)} - z^2(u-1) E_d^{(k_1+k)}$$

Then from Lemma 12:

$$\text{tr}(\alpha e_{d,3}) = z x_{k_1} E_d^{(k)} + z(u-1) \frac{\left[E_d^{(k+k_1)} \right]^2}{x_{k+k_1}} - z^2(u-1) E_d^{(k_1+k)}.$$

Applying now the E -condition to X_d yields immediately $\text{tr}(\alpha e_{d,3}) = \text{tr}(\alpha)\text{tr}(e_{d,3})$.

4.3. Solutions of the E -system. The E -system has always a not-all-zero solution. For example, we have the cyclic solution $x_k = \zeta^k$, where ζ is a primitive d th root of unity. Indeed:

$$E_d^{(m)} = \sum_{s=0}^{d-1} x_{m+s} x_{d-s} = \zeta^m \Rightarrow E_d = 1 \text{ and } x_m E_d = x_m = \zeta^m$$

The solution $x_k = \zeta^k$ of the E -system is a special case of Lemma 11, so it is not interesting for our topological purposes. It is worth observing at this point that the values $x_i = c^i$ of Lemma 11 do not comprise, in general, a solution of the E -system. For example, for $d = 3$ we have the E -system:

$$\begin{aligned} x_1 + x_2^2 &= 2x_1^2 x_2 \\ x_1^2 + x_2 &= 2x_1 x_2^2 \end{aligned}$$

Substituting now $x_i = c^i$ does not automatically satisfy the system equations.

Beyond the above cyclic solution, for $d = 3, 4$ and 5 we run the Mathematica program and we found other solutions of the E -system for which

$$\text{tr}(e_{d,i}) \neq 1, \text{ for all } i$$

For example, in the case $d = 3$ we have the non-trivial solutions:

$$x_1 = x_2 = -\frac{1}{2} \quad \text{or} \quad x_1 = \frac{1}{2}(-1 + i\sqrt{3}), \quad x_2 = \frac{1}{2}(1 + i\sqrt{3})$$

and also the solution where we take the conjugates in the previous one.

Consider now the real numbers

$$\delta_i := \frac{-(-1)^{i(d-1)}}{d-1} \quad (i = 1, \dots, d-1)$$

and denote $E_d^{(m)}(\delta)$ the evaluation of $E_d^{(m)}$ at $x_i = \delta_i$. According to (4.1) we have $E_d^{(0)} = 1 + \sum_{s=1}^{d-1} x_s x_{d-s}$, then

$$E_d^{(0)}(\delta) = 1 + \sum_{s=1}^{d-1} \frac{(-1)^{s(d-1)}(-1)^{(d-s)(d-1)}}{(d-1)^2} = 1 + \sum_{s=1}^{d-1} \frac{(-1)^{d(d-1)}}{(d-1)^2}$$

therefore

$$E_d^{(0)}(\delta) = 1 + \frac{(d-1)}{(d-1)^2} = \frac{d}{d-1}$$

Proposition 4. $\delta = \{\delta_i\}$ is a solution of the E -system (4.2).

Proof. For $m = 1, \dots, d-1$ we have:

$$E_d^{(m)} = 2x_m + \sum_{s \neq 0, d-m} x_{m+s} x_{d-s}$$

So:

$$\begin{aligned}
E_d^{(m)}(\delta) &= \frac{-2(-1)^{m(d-1)}}{d-1} + \frac{1}{(d-1)^2} \sum_{s \neq 0, d-m} (-1)^{(d+m)(d-1)} \\
&= \frac{-2(-1)^{m(d-1)}}{d-1} + \frac{(-1)^{(d-1)m}}{(d-1)^2} \sum_{s \neq 0, d-m} (-1)^{d(d-1)} \\
&= \frac{-2(-1)^{m(d-1)}}{d-1} + \frac{(-1)^{(d-1)m}}{(d-1)^2} (d-2) \\
&= \frac{-2(-1)^{m(d-1)}(d-1) + (-1)^{(d-1)m}(d-2)}{(d-1)^2} \\
&= \frac{-d(-1)^{m(d-1)}}{(d-1)^2} = \delta_m E_d^{(0)}(\delta).
\end{aligned}$$

□

Remark 6. Note that for the solution $\delta = \{\delta_i\}$ of the E -system we have $\text{tr}(e_{d,i}) = \frac{1}{d} E_d^{(0)}(\delta) = \frac{1}{d-1} \neq 1$ for $d \neq 2$.

In the Appendix to this paper we give the general solution of the E -system, due to Paul Gérardin.

4.4. Lifting solutions to the p -adic level. Let r and s be two positive integers, such that $r \geq s$. We prove now that a solution of the E -system for $d = p^s$ lifts to a solution of the corresponding E -system for $d' = p^r$. This is important for showing that there are also interesting solutions at the p -adic level.

Given $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d$, we define $\zeta' = (\zeta'_1, \dots, \zeta'_{d'}) \in \mathbb{C}^{d'}$ as follows:

$$\zeta'_j = \begin{cases} \zeta_i, & \text{for } i = 1, \dots, d-1 \\ \zeta_j, & \text{for } i \equiv j \pmod{d}. \end{cases}$$

Then we have the following commutative diagram:

$$\begin{array}{ccc}
R[\mathfrak{X}_r] & \xrightarrow{\delta_s^r} & R[\mathfrak{X}_s] \\
& \searrow \text{ev}_{\zeta'} & \downarrow \text{ev}_{\zeta} \\
& & R
\end{array}
\tag{4.3}$$

where ev_c is the evaluation homomorphism at $c \in \mathbb{C}^m$.

Proposition 5. Let $d = p^s$ and $d' = p^r$ with $r \geq s$. If ζ is a solution of the system of equations $E_d^{(k)} = x_k E_d$ ($k = 1, \dots, d-1$) then ζ' is a solution of the system of equations $E_{d'}^{(k)} = x_k E_{d'}$ ($k = 1, \dots, d'-1$).

Proof. The equation $E_d^{(k)} = x_k E_d$ can be written as: $\text{tr}_d(e_{d,n}^{(k)}) = \text{tr}_d(e_{d,n} t_{n+2}^k)$.

Now: $\text{ev}_{\zeta'}(\text{tr}_{d'}(e_{d',n}^{(k)})) = (\text{ev}_{\zeta'} \circ \text{tr}_{d'})(e_{d',n}^{(k)})$, and from diagram (4.3) we have:

$$\begin{aligned}
 \text{ev}_{\zeta'}(\text{tr}_{d'}(e_{d',n}^{(k)})) &= (\text{ev}_{\zeta} \circ \delta_s^r \circ \text{tr}_{d'})(e_{d',n}^{(k)}) \\
 &= (\text{ev}_{\zeta} \circ \text{tr}_d \circ \rho_d^{d'})(e_{d',n}^{(k)}) \quad (\text{Lemma 7}) \\
 &= \text{ev}_{\zeta}(\text{tr}_d(\rho_d^{d'}(e_{d',n}^{(k)}))) \\
 &= \text{ev}_{\zeta}(\text{tr}_d(e_{d,n}^{(k)})) \\
 &= \text{ev}_{\zeta}(\text{tr}_d(e_{d,n} t_{n+2}^k)) \quad (\text{Induction hypothesis}) \\
 &= \text{ev}_{\zeta}(\text{tr}_d(\rho_d^{d'}(e_{d',n} t_{n+2}^k))) \\
 &= \text{ev}_{\zeta}(\delta_s^r(\text{tr}_{d'}(e_{d',n} t_{n+2}^k))) \quad (\text{Lemma 7}) \\
 &= \text{ev}_{\zeta'}(\text{tr}_{d'}(e_{d',n} t_{n+2}^k)) \quad (\text{Diagram (4.3)}).
 \end{aligned}$$

Hence ζ' is a solution of the system: $E_{d'}^{(k)} = x_k E_{d'}$ ($1 \leq k \leq d' - 1$). \square

5. ISOTOPY INVARIANTS OF FRAMED AND p -ADIC FRAMED LINKS

The aim of this section is to define an infinite family of isotopy invariants of oriented framed links, using the Markov traces of Theorems 3 and 4 and the Markov equivalence of Theorem 5. Also, to define an isotopy invariant of p -adic oriented framed links, using the p -adic Markov trace of Theorem 4 and the p -adic Markov equivalence of Theorem 5. (*Isotopy* is the notion of topological equivalence for knots and links).

5.1. *Framed link invariants from tr_d .* Using Theorem 6 and Lemma 8 we can proceed with the factorization of $\text{tr}(\alpha e_{d,n})$, assuming the E -condition. Indeed, we then have:

$$\begin{aligned}
 \text{tr}_d(\alpha g_n^{-1}) &= (z - (u^{-1} - 1)E_d + (u^{-1} - 1)z) \text{tr}_d(\alpha) \\
 &= \frac{z + (u - 1)E_d}{u} \text{tr}_d(\alpha) = \text{tr}_d(g_n^{-1}) \text{tr}_d(\alpha),
 \end{aligned}$$

where E_d was defined in (3.4).

In analogy to the construction of the Jones polynomial[2], we will do first a re-scaling, by which αg_n and αg_n^{-1} will be assigned the same trace value for any $\alpha \in Y_{d,n}(u)$. More precisely, we define

$$\omega := \frac{z + (u - 1)E_d}{uz}$$

so $\text{tr}_d(g_n^{-1}) = \omega z$. Then the map:

$$\sigma_i \mapsto \sqrt{\omega} g_i, \quad t_j^{m(\bmod d)} \mapsto t_j^{m(\bmod d)}$$

defines the representation:

$$\Omega_{d,\omega} : \mathcal{F}_{d,n} \longrightarrow Y_{d,n}(u)$$

Moreover, composing with the natural projection from \mathcal{F}_n to $\mathcal{F}_{d,n}$, the representation $\Omega_{d,\omega}$ lifts to a representation of \mathcal{F}_n to $Y_{d,n}(u)$, and we retain for this the same notation, $\Omega_{d,\omega}$. Then we have the following:

Definition 10. For any framed braid $\alpha \in \mathcal{F}_n$ we define for its closure $\widehat{\alpha}$:

$$\Gamma_d(\widehat{\alpha}) := \left(\frac{1 - \omega u}{\sqrt{\omega}(1 - u)E_d} \right)^{n-1} (\text{tr}_d \circ \Omega_{d,\omega})(\alpha)$$

Defining further the *exponent* $\epsilon(\alpha)$ of α as the algebraic sum of the exponents of the σ_i 's in α and denoting

$$\Delta := \frac{1 - \omega u}{\sqrt{\omega}(1 - u)E_d} = \frac{1}{z\sqrt{\omega}}$$

we have:

$$\Gamma_d(\widehat{\alpha}) = \Delta^{n-1}(\sqrt{\omega})^{\epsilon(\alpha)} \text{tr}_d(\alpha)$$

Let now \mathcal{L} denote the set of oriented framed links and let $\mathbb{C}(z, x_1, \dots, x_d)$ be, as usual, the ring of rational functions on z, X_d with complex coefficients. Then we have the following.

Theorem 7. *If the set X_d satisfies the E-condition then the map Γ_d is an isotopy invariant of oriented framed links.*

$$\begin{array}{ccc} \Gamma_d & : & \mathcal{L} \longrightarrow \mathbb{C}(z, x_1, \dots, x_d) \\ & & \widehat{\alpha} \mapsto \Gamma_d(\widehat{\alpha}) \end{array}$$

Proof. By the classical Alexander theorem, any link can be isotoped to the closure of some braid. Thus, showing that Γ_d is constant on the isotopy class of the oriented framed link $\widehat{\alpha}$ for any $\alpha \in \mathcal{F}_\infty$ implies the statement. By Theorem 5 isotopy classes of oriented framed links are in bijection with equivalence classes of braids. Hence, we must prove that $\Gamma_d(\widehat{\alpha}) = \Gamma_d(\widehat{\alpha\sigma_n}) = \Gamma_d(\widehat{\alpha\sigma_n^{-1}})$, for all $\alpha \in \mathcal{F}_n$. The first equality is basically taken care by the coefficient of Δ in Definition 10, the second by the re-scaling of the trace. More precisely, and since $\epsilon(\alpha g_n) = \epsilon(\alpha) + 1$ and $\epsilon(\alpha g_n^{-1}) = \epsilon(\alpha) - 1$, we obtain:

$$\Gamma_d(\widehat{\alpha\sigma_n}) = \Delta^n \sqrt{\omega}^{\epsilon(\alpha g_n)} \text{tr}_d(\alpha g_n) = \Delta \sqrt{\omega} z \Gamma_d(\widehat{\alpha})$$

$$\Gamma_d(\widehat{\alpha\sigma_n^{-1}}) = \Delta^n \sqrt{\omega}^{\epsilon(\alpha g_n^{-1})} \text{tr}_d(\alpha g_n^{-1}) = \Delta \sqrt{\omega} z \Gamma_d(\widehat{\alpha})$$

Therefore, and since $\Delta \sqrt{\omega} z = 1$, the proof of the Theorem is concluded. \square

Remark 7. For any oriented link with all framings zero, it follows by Remark 3, that the invariant Γ_d coincide with the HOMFLYPT (2-variable Jones) polynomial[2] for oriented classical links.

5.2. *Some computations.* Clearly, for the unknot O with framing zero we have $\Gamma_d(O) = 1$.

- For the unknot O^k with framing $k \in \mathbb{Z}$ we have $\Gamma_d(O^k) = x_k$ where $\{x_1, \dots, x_{d-1}\}$ constitutes a solution of the E -system.
- Let $H = \widehat{\sigma_1^2 t_1^k t_2^l}$ be the Hopf link with framings $k, l \in \mathbb{Z}$. We have $\epsilon(\sigma_1^2 t_1^k t_2^l) = 2$ and, using Remark 5, we find:

$$\Gamma_d(H) = \Delta \omega \operatorname{tr}_d(g_1^2 t_1^k t_2^l) = \Delta \omega \left[x_l x_k + (u-1) E_d^{(k+l)} - (u-1) z x_{k+l} \right]$$

where $\{x_1, \dots, x_{d-1}\}$ constitutes a solution of the E -system.

- Let $T = \widehat{\sigma_1^3 t_1^k}$ be the right-handed trefoil with framing $k \in \mathbb{Z}$. We have $\epsilon(\sigma_1^3 t_1^k) = 3$ and, using Lemma 1 and Remark 5, we find:

$$\Gamma_d(T) = \Delta \sqrt{\omega}^3 \operatorname{tr}_d(g_1^3 t_1^k) = \Delta \sqrt{\omega}^3 \left[(u^2 - u + 1) z x_k - u(u-1) E_d^{(k)} \right]$$

where $\{x_1, \dots, x_{d-1}\}$ constitutes a solution of the E -system.

- Let, finally, $T' = \widehat{\sigma_1^{-3} t_1^k}$ be the left-handed trefoil with framing $k \in \mathbb{Z}$. We have $\epsilon(\sigma_1^{-3} t_1^k) = -3$ and, using Lemma 1 and Remark 5, we find:

$$\Gamma_d(T') = \Delta \sqrt{\omega}^{-3} \left[(u^{-3} - u^{-2} + u^{-1}) z x_k - (u^{-3} - u^{-2} + u^{-1} - 1) E_d^{(k)} \right]$$

where $\{x_1, \dots, x_{d-1}\}$ constitutes a solution of the E -system.

5.3. *A skein relation for Γ_d .* Let L_+ , L_- , L_m and L'_m , $m = 0, \dots, d-1$, be diagrams of oriented framed links, which are all identical, except near one crossing, where they differ by the ways indicated in Figure 4. Then we have the following result.

Proposition 6. *The invariant Γ_d satisfies the following skein relation:*

$$\frac{1}{\sqrt{\omega}} \Gamma_d(L_+) - \sqrt{\omega} \Gamma_d(L_-) = \frac{u^{-1} - 1}{d} \sum_{m=0}^{d-1} \Gamma_d(L_m) - \frac{u^{-1} - 1}{d \sqrt{\omega}} \sum_{m=0}^{d-1} \Gamma_d(L'_m)$$

The above linear skein relation arises from Eq. (1.5) and is diagrammatically related to Figure 4, but with different coefficients.

Proof. The proof is standard. By the Alexander theorem for framed links we may assume that L_+ is in braided form and that $L_+ = \widehat{\beta \sigma_i}$ for some $\beta \in \mathcal{F}_n$. Also that $L_- = \widehat{\beta \sigma_i^{-1}}$. Recall now (1.3) and apply relation (1.5) for the g_i^{-1} in the expression:

$$\Gamma_d(L_-) = \Delta^{n-1} (\sqrt{\omega})^{\epsilon(\beta g_i^{-1})} \operatorname{tr}_d(\beta g_i^{-1})$$

Finally, noting that $\epsilon(\beta g_i^{-1}) = \epsilon(\beta) - 1$, $\epsilon(\beta g_i) = \epsilon(\beta) + 1$, $\epsilon(\beta t_i^m t_{i+1}^{-m}) = \epsilon(\beta)$ and $\epsilon(\beta t_i^m t_{i+1}^{-m} g_i) = \epsilon(\beta) + 1$ we obtain the stated relation. \square

5.4. Framed and p -adic framed link invariants from τ_{p^∞} . The aim of this subsection is to extend the values of the invariant Γ_d to the p -adic context. Let \mathcal{L}_{p^∞} denote the set of oriented p -adic framed links. For positive integer r, s such that $r \geq s$, the connecting ring epimorphism δ_s^r (recall (3.3)) yields a connecting epimorphism Ξ_s^r from the ring of rational functions $\mathbb{C}(z, \mathfrak{X}_r)$ to the ring of rational functions $\mathbb{C}(z, \mathfrak{X}_s)$. It is a routine to prove the following lemma.

Lemma 13. *For all $r \geq s \geq v$, the following diagram is commutative.*

$$\begin{array}{ccccccc}
 \cdots & \longleftarrow & \mathcal{L}_{p^\infty} & \xleftarrow{\text{Id}} & \mathcal{L}_{p^\infty} & \xleftarrow{\text{Id}} & \mathcal{L}_{p^\infty} \longleftarrow \cdots \\
 & & \downarrow \Gamma_{p^v} & & \downarrow \Gamma_{p^s} & & \downarrow \Gamma_{p^r} \\
 \cdots & \longleftarrow & \mathbb{C}(z, \mathfrak{X}_v) & \xleftarrow{\Xi_v^s} & \mathbb{C}(z, \mathfrak{X}_s) & \xleftarrow{\delta_s^r} & \mathbb{C}(z, \mathfrak{X}_r) \longleftarrow \cdots
 \end{array}$$

The ring $\varprojlim R[\mathfrak{X}_r]$ turns out to be an integral domain. We shall also denote R_{p^∞} the field of fractions of $\varprojlim R[\mathfrak{X}_r]$. Taking now inverse limits in the diagram of Lemma 13 we obtain the map:

$$\Gamma_{p^\infty} := \varprojlim_{r \in \mathbb{N}} \Gamma_{p^r}$$

Theorem 8. *If for all r the set \mathfrak{X}_r satisfies the E -condition, then the map*

$$\begin{array}{ccc}
 \Gamma_{p^\infty} & : & \mathcal{L}_{p^\infty} \longrightarrow R_{p^\infty} \\
 & & \widehat{\underline{\alpha}} \longmapsto (\Gamma_p(\widehat{\alpha_1}), \Gamma_{p^2}(\widehat{\alpha_2}), \dots)
 \end{array}$$

for any $\underline{\alpha} = (\alpha_r)_r \in \varinjlim_n \mathcal{F}_{p^\infty, n}$ is constant on the equivalence classes defined by the p -adic version of Theorem 5. Moreover,

$$\Gamma_{p^\infty}(\widehat{\alpha}) = \left(\frac{1 - \omega u}{\sqrt{\omega}(1 - u)E_{p^r}} \right)^{n-1} (\sqrt{\omega})^{\epsilon(\alpha)} \tau_{p^\infty}(\alpha) = \Delta^{n-1} (\sqrt{\omega})^{\epsilon(\alpha)} \tau_{p^\infty}(\alpha)$$

for some r and where $\alpha \in \mathcal{F}_n$.

Proof. By Proposition 5 we have non-trivial solutions of the E -system in the p -adic context. Let now $\underline{\beta} = (\beta_r)_r$ and $\underline{\alpha} = (\alpha_r)_r \in \mathcal{F}_{p^\infty, \infty}$ be Markov equivalent p -adic framed braids. Then, according to Theorem 5, we have that in each position the modular framed braid β_r is Markov equivalent to the braid α_r . So, $\Gamma_{p^r}(\widehat{\beta_r}) = \Gamma_{p^r}(\widehat{\alpha_r})$, hence $\Gamma_{p^\infty}(\widehat{\underline{\alpha}}) = \Gamma_{p^\infty}(\widehat{\underline{\beta}})$. Moreover, restricting Γ_{p^∞} to the set \mathcal{L} of classical oriented framed links we have that Γ_{p^∞} is also an isotopy invariant of oriented framed links. Note that, given a framed braid $\alpha \in \mathcal{F}_n$, after some position the entries $\Gamma_{p^r}(\widehat{\alpha})$ of $\Gamma_{p^\infty}(\widehat{\alpha})$ will all have the same formal expression. So, and by Subsection 4.4, the value $\Gamma_{p^\infty}(\widehat{\alpha})$ can be seen

as a constant sequence in R_{p^∞} , that is, an element in some $\mathbb{C}(z, \mathfrak{X}_r)$. So, we have:

$$\begin{aligned} \Gamma_{p^\infty}(\widehat{\alpha}) &= (\Gamma_p(\widehat{\alpha}), \Gamma_{p^2}(\widehat{\alpha}), \dots) \\ &= (\Delta^{n-1}(\sqrt{\omega})^{\epsilon(\alpha)} \tau_1(\alpha_1), \Delta^{n-1}(\sqrt{\omega})^{\epsilon(\alpha)} \tau_2(\alpha), \dots) \\ &= \Delta^{n-1}(\sqrt{\omega})^{\epsilon(\alpha)} (\tau_1(\alpha), \tau_2(\alpha), \dots) \\ &= \Delta^{n-1}(\sqrt{\omega})^{\epsilon(\alpha)} \tau_{p^\infty}(\alpha) \end{aligned}$$

□

6. APPENDIX: THE E -SYSTEM BY PAUL GÉRARDIN

We interpret each polynomial in (4.1)

$$\sum_{0 \leq s < d} x_{m+s} x_{d-s}, \quad 0 \leq m < d$$

in the d complex numbers x_0, x_1, \dots, x_{d-1} as the value at m of the convolution product by itself of the element $x : s \mapsto x_s$ in the complex algebra $\mathbb{C}[D]$ of the cyclic group $D = \mathbb{Z}/d\mathbb{Z}$: the convolution product $f * g$ of two elements $f, g \in \mathbb{C}[D]$ is

$$f * g : w \mapsto \sum_{u+v=w} f(u)g(v),$$

the sum being on the set of $(u, v) \in D \times D$ with sum w .

The algebra $\mathbb{C}[D]$ is commutative algebra with unit δ_0 , the characteristic function of the unit element $0 \in D$. It is the direct sum of its simple ideals $\mathbb{C}e_a, a \in D$, the e_a 's being the characters of the group D :

$$e_a : u \mapsto e^{2\pi i a u / d}.$$

They satisfy the following relations : $e_a * e_b$ is de_a for $a = b$ and 0 otherwise, so that the $e_a/d, a \in D$ are its elementary idempotents.

The algebra $\mathbb{C}[D]$ has another product, with unit e_0 , given by the product of values:

$$fg : w \mapsto f(w)g(w),$$

and is the direct sum of its simple ideals $\mathbb{C}\delta_a, a \in D$, where δ_a is the characteristic function of the element $a \in D$; they are also the elementary idempotents for this structure.

The Fourier transform on $\mathbb{C}[D]$:

$$f \mapsto \widehat{f} : v \mapsto (f * e_v)(0) = \sum_{u \in D} f(u) e_v(-u)$$

is a linear automorphism. In particular, $\widehat{\delta}_a = e_{-a}$, $\widehat{e}_a = d\delta_a$, $a \in D$. Its inverse is $f \mapsto (u \mapsto \frac{1}{d}\widehat{f}(-u))$, which means $\widehat{\widehat{f}}(u) = d f(-u)$

The Fourier transform sends the convolution product to the product of values :

$$\widehat{f * g} = \widehat{f} \widehat{g},$$

hence $\widehat{fg} = d^{-1}\widehat{f} * \widehat{g}$.

The E -condition (4.2) can now be written as

$$x * x = (x * x)(0) x.$$

To solve the E -system $x(0) = 1$, $x * x = (x * x)(0) x$, we use Fourier transform, to get

$$x(0) = 1, \widehat{x}^2 = (x * x)(0) \widehat{x}$$

If $(x * x)(0) = 0$, then $\widehat{x}^2 = 0$ so \widehat{x} is 0 and also is x , which is excluded by the condition $x(0) = 1$. Now, the equation says that the function \widehat{x} is constant on its support S where it is $(x * x)(0)$. As the characteristic function of S is, up to the factor d , the Fourier transform of the sum of e_a , $a \in S$, we have shown that

$$x = (x * x)(0) \frac{1}{d} \sum_{s \in S} e_s$$

As $x(0) = 1$, we have $(x * x)(0) \frac{1}{d} |S| = 1$, with $|S|$ the cardinality of S .

Finally, we have proved that the solutions of the E -system are the functions x_S parametrized by the non-empty subsets S of the cyclic group D of order d as follows:

$$x_S = \frac{1}{|S|} \sum_{s \in S} e_s$$

For $S = D$, it is the trivial solution δ_0 . The complement of the support of any non trivial solution is another solution. In particular, each element $a \in D$ defines two solutions of the E -system : one is the character e_a , the other is given by $\frac{e_a}{1-d}$ outside 0. When the order d is even, we can take $a = d/2$, this

gives the solution $u \mapsto \frac{(-1)^u}{1-d}$, $u \neq 0$.

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